

Cellular approximations of fusion systems

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Abstract

In this paper we study the cellularization of classifying spaces of saturated fusion systems over finite p -groups with respect to classifying spaces of finite p -groups. We give explicit criteria to decide when a classifying space is cellular and we explicitly compute the cellularization for a family of exotic examples.

1 Introduction

The transfer plays an important role both in stable homotopy theory of classifying spaces and group cohomology. Given a finite group G and a prime p , if S is a Sylow p -subgroup, the properties of the transfer imply that the mod p cohomology of G injects into the mod p cohomology of S . In stable homotopy theory, the spectrum of BG_p^\wedge is a retract of the spectrum of BS and the splitting is constructed by an idempotent in stable selfmaps of the spectrum of BS .

In 1990s, E. Dror-Farjoun and W. Chachólski generalized the concept of CW-complex, spaces build from spheres by means pointed homotopy colimits. Let A be a pointed space and let $C(A)$ denote the smallest collection of pointed spaces that contains A and it is closed by weak equivalences and pointed homotopy colimits. A pointed space X is A -cellular if $X \in C(A)$. These concepts are also defined in stable homotopy category.

In the stable context, the fact that BG_p^\wedge is a stable retract of BS implies that that $\Sigma_+^\infty BG_p^\wedge$ belongs to $C(\Sigma_+^\infty BS)$. In unstable homotopy theory of classifying spaces, BG_p^\wedge is not a retract of BS , but we can ask ourselves whether BG_p^\wedge is in the cellular class $C(BS)$, or more generally, given a finite p -group P , $BG_p^\wedge \in C(BP)$?

The homotopy type of BG_p^\wedge is determined by the p -local structure of G , the fusion system associated to G . Given a finite p -group S , p a prime, a *fusion system* over S is a subcategory of the category of groups whose objects are the subgroup of S and morphisms are given a set of injective homomorphisms, containing those which are induced by conjugation by elements of S . A fusion system \mathcal{F} is *saturated* if it verifies certain axioms such as would be holded if S were a Sylow p -subgroup of a finite group. These ideas were developed by L. Puig in an unpublished notes. Afterwards, D. Benson suggested the idea of associating a

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“classifying space” to each saturated fusion system (see [Ben98]). The notion of classifying space was formulated by C. Broto, R. Levi and B. Oliver in [BLO03b], where the notion of “centric linking system” (or “ p -local finite group”) associated to saturated fusion systems appears. At that time, it was not known if every saturated fusion has an associated linking system.

K. Ragnarsson [Rag06] constructed a classifying space spectrum $B\mathcal{F}$ associated to \mathcal{F} by splitting the spectrum of BS via an idempotent stable selfmap. Analogously to the situation for finite groups, we have then $B\mathcal{F} \in C(BS)$.

Recently, A. Chermak [Che13] has proved the existence and uniqueness of centric linking systems, that means, each saturated fusion system \mathcal{F} has a unique (up to isomorphism) centric linking system associated to \mathcal{F} , and so a unique (up to homotopy equivalence) classifying space $B\mathcal{F}$. See Section 2 for specific definitions, details and main results which we will use in the rest of the paper about fusion systems.

Previous works in finite groups (see [Flo07], [FS07] and [FF11]) suggest the strong relationship between the cellularity properties of BG_p^\wedge with respect to classifying spaces of finite p -groups and the fusion structure of G at the prime p .

Let P be a finite p -group, we will denote by $Cl_{\mathcal{F}}(P)$ the smallest strongly \mathcal{F} -closed subgroup in S which contains all the images of homomorphisms $P \rightarrow S$ (see Section 3 below). The main result of this paper is the following theorem.

Theorem 5.1. *Let \mathcal{F} be a saturated fusion system over a finite p -group S and let P be a finite p -group. Then $B\mathcal{F}$ is BP -cellular if and only if $S = Cl_{\mathcal{F}}(P)$.*

Corollary 5.2. *Let (S, \mathcal{F}) be a saturated fusion system and P a finite p -group.*

- (a) *The classifying space $B\mathcal{F}$ is BS -cellular.*
- (b) *Let (S, \mathcal{F}') be a saturated fusion system with $\mathcal{F} \subset \mathcal{F}'$. If $B\mathcal{F}$ is BP -cellular then $B\mathcal{F}'$ is also BP -cellular.*
- (c) *Let A be a pointed connected space. If $Cl_{\mathcal{F}}((\pi_1 A)_{ab}) = S$, then $B\mathcal{F}$ is A -cellular.*
- (d) *Let $\Omega_{p^m}(S)$ be the (normal) subgroup of S generated by its elements of order p^i , which $i \leq m$. Then $B\mathcal{F}$ is $B\mathbb{Z}/p^m$ -cellular if and only if $S = Cl_{\mathcal{F}}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $B\mathbb{Z}/p^m$ -cellular for all $m \geq m_0$.*

There exists an augmented idempotent endofunctor $CW_A: Spaces_* \rightarrow Spaces_*$ such that for all pointed space X , the space $CW_A X$ is A -cellular and the augmentation map $c_X: CW_A X \rightarrow X$ is an A -equivalence, that means, it is induced a weak equivalence in pointed mapping space $(c_X)_*: \text{map}_*(A, CW_A X) \xrightarrow{\cong} \text{map}_*(A, X)$. Roughly speaking, $CW_A X$ is the best A -cellular approximation of X . We will say that $CW_A X$ is the A -cellularization of X and the map $c_X: CW_A X \rightarrow X$ is the A -cellular approximation of X . See [DF96] for more details about the construction and main properties of the functor CW_A .

The strategy of proof of Theorem 5.1 is the analysis of the Chachólski fibration to compute $CW_{BP}(B\mathcal{F})$ which is described in [Cha96]. Let C be the homotopy cofibre of the evaluation map $ev: \bigvee_{[BP, B\mathcal{F}]_*} BP \rightarrow B\mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is the homotopy fibre of the

composite $r: B\mathcal{F} \rightarrow C \rightarrow P_{\Sigma BP}C$, where $P_{\Sigma BP}$ denotes the ΣBP -nullification functor defined by A. K. Bousfield in [Bou94]. We prove that $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}$ if and only if the map r_p^\wedge is null-homotopic (see the proof of Theorem 5.1). Therefore, the BP -cellularity of $B\mathcal{F}$ is equivalent to the homotopy nullity of the map r_p^\wedge .

To approach this question, we study the kernel of r_p^\wedge in the sense of D. Notbohm introduced in [Not94] for maps from classifying spaces of compact Lie groups (Section 3). Given a map $f: B\mathcal{F} \rightarrow Z$, where Z is a connected p -complete and $\Sigma B\mathbb{Z}/p$ -null space (that is, $Z_p^\wedge \simeq Z$ and $\text{map}_*(\Sigma B\mathbb{Z}/p, Z) \simeq *$), then $\ker(f) := \{x \in S \mid f|_{B\langle x \rangle} \simeq *\}$. We show that $\ker(f)$ is a strongly \mathcal{F} -closed subgroup of S (Proposition 3.5) with the following main property.

Theorem 3.6. *Let (S, \mathcal{F}) be a saturated fusion system. Let Z be a connected p -complete and $\Sigma B\mathbb{Z}/p$ -null space. A map $f: B\mathcal{F} \rightarrow Z$ is null-homotopic if and only if $\ker(f) = S$.*

Furthermore, any strongly \mathcal{F} -closed subgroup $K \leq S$ is the kernel of a map from $B\mathcal{F}$ to $B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ for certain $m \geq 0$ (Proposition 3.7).

In Section 4 we study homotopy properties of $CW_{BP}(B\mathcal{F})$ needed for the proof of the main theorem. Section 5 contains the proof of Theorem 5.1. A key step is the computation of the kernel of r_p^\wedge .

Proposition 5.5. *Let (S, \mathcal{F}) be a saturated fusion system. Then $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$.*

The last two sections are devoted to give explicit examples. Concretely, in Section 6 we describe a strategy to compute the BP -cellularization of $B\mathcal{F}$ when $S \neq Cl_{\mathcal{F}}(P)$. This is the case when we have a homotopy factorization of $r_p^\wedge: B\mathcal{F} \rightarrow (P_{\Sigma BP}C)_p^\wedge$ by a map $\tilde{r}_p^\wedge: B\mathcal{F}' \rightarrow (P_{\Sigma BP}C)_p^\wedge$ with trivial kernel (verifying certain technical conditions in Proposition 6.2). This is the case when $Cl_{\mathcal{F}}(S)$ is normal in \mathcal{F} .

Corollary 6.5. *Let (S, \mathcal{F}) be a fusion system and let P be a finite p -group. If $Cl_{\mathcal{F}}(P) \triangleleft \mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is homotopy equivalent to the homotopy fibre of $B\mathcal{F} \rightarrow B(\mathcal{F}/Cl_{\mathcal{F}}(P))$.*

This result allow us compute, for all $r \geq 1$, the $B\mathbb{Z}/p^r$ -cellularization of the classifying space of $\mathbb{Z}/p^n \wr \mathbb{Z}/q$, with $p \neq q$, and of the Suzuki group $Sz(2^n)$, with n an odd integer at least 3 (Example 6.6).

The last section contains an explicit description of the $B\mathbb{Z}/3^l$ -cellularization of the classifying space of a family of exotic fusion systems over a finite 3-group given in [DRV07].

Corollary 7.1. *Let \mathcal{F} be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ such that \mathcal{F} has at least one \mathcal{F} -Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then*

- (i) *If $\gamma = 0$, then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$ -cellular for all $l \geq 1$.*
- (ii) *Assume $\gamma \neq 0$. Then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$ -cellular if and only if $l \geq 2$. If $l = 1$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = \langle s, s_2 \rangle$.*

Moreover, when $\gamma \neq 0$ and $l = 1$, we show that $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ is the homotpy fibre of a map $B\mathcal{F} \rightarrow (B\Sigma_3)_3^\wedge$.

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2 Preliminaries on the homotopy theory of fusion systems

A saturated fusion system is a small subcategory of the category of groups which encodes fusion/conjugacy data between subgroups of a fixed finite p -group S , as formalized by L. Puig (see [Pui06] and also [AKO11]). Such objects have classifying spaces which satisfy many of the rigid homotopy theoretic properties of p -completed classifying spaces of finite groups, as generalized by C. Broto, R. Levi and B. Oliver (see [BLO03b]).

Definition 2.1. Let S be a finite p -group. A *saturated fusion system on S* is a subcategory \mathcal{F} of the category of groups with $\text{Ob}(\mathcal{F})$ the set of all subgroups of S and such that it satisfies the following properties. For all $P, Q \leq S$:

- (f.1) $\text{Hom}_S(P, Q) \subset \text{Hom}_{\mathcal{F}}(P, Q) \subset \text{Inj}(P, Q)$; and
- (f.2) each $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ is the composite of an isomorphism in \mathcal{F} followed by an inclusion.

For all $P \leq S$ and all $P' \leq S$ which is \mathcal{F} -conjugate to P (P and P' are isomorphic as objects in \mathcal{F}):

- (s.1) For all $P \leq S$ which is *fully normalized in \mathcal{F}* (i.e. $|N_S(P)| \geq |N_S(P')|$ for all $P' \mathcal{F}$ -conjugate to P), P is also *fully centralized in \mathcal{F}* ($|C_S(P)| \geq |C_S(P')|$ for all $P' \mathcal{F}$ -conjugate to P), and $\text{Out}_S(P) \in \text{Syl}_p(\text{Out}_{\mathcal{F}}(P))$.
- (s.2) Let $P \leq S$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$ be such that $\varphi(P)$ is fully centralized. If we set

$$N_{\varphi} = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi(P))\},$$

then there is $\bar{\varphi} \in \text{Hom}_{\mathcal{F}}(N_{\varphi}, S)$ such that $\bar{\varphi}|_P = \varphi$.

The standard example is given by the fusion category of a finite group G . Given a finite group G with a fixed Sylow p -subgroup S , let $\mathcal{F}_S(G)$ be the category with $\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$ for all $P, Q \leq S$, where $\text{Hom}_G(P, Q) = \{\varphi \in \text{Hom}(P, Q) \mid \varphi = c_g \text{ for some } g \in G\}$. This category $\mathcal{F}_S(G)$ satisfies the saturation axioms (see [BLO03b, Proposition 1.3]).

In order to recover the homotopy type of the Bousfiel-Kan p -completion of the classifying space BG_p^{\wedge} , Broto-Levi-Oliver [BLO03a] introduce a new category defined using the group G . A p -subgroup $P \leq G$ is p -centric if $Z(P')$ is a Sylow p -subgroup of $C_G(P')$ for all $P' G$ -conjugate to P . Let $\mathcal{L}_S(G)$ be the category whose objects are p -centric subgroups of G with $\text{Mor}_{\mathcal{L}_S(G)}(P, Q) = \{x \in G \mid xP^{-1}x \leq Q\} / O^p(C_G(P))$ for all $P, Q \leq S$. In [BLO03a] the authors proved that the classifying space of this p -local structure is $|\mathcal{L}_S(G)|_p^{\wedge} \simeq BG_p^{\wedge}$. This new structure one can associate to a finite group can also be generalized in the context of abstract fusion systems.

Definition 2.2. [BLO03b] Let \mathcal{F} be a saturated fusion system over a finite p -group S . A *centric linking system associated to \mathcal{F}* is a category \mathcal{L} whose objects are the \mathcal{F} -centric subgroups of S (i.e., the subgroups $P \leq S$ such that $C_S(P) = Z(P)$), together with a functor $\pi: \mathcal{L} \rightarrow \mathcal{F}^c$, and “distinguished” monomorphisms $\delta_P: P \rightarrow \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$, which satisfies the following conditions:

- (I.1) π is the identity on objects and surjective on morphisms. For each pair of objects $P, Q \leq \mathcal{L}$, $Z(P)$ acts freely on $\text{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \text{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\text{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

- (I.2) For each \mathcal{F} -centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \text{Aut}_{\mathcal{L}}(P)$ to $c_g \in \text{Aut}_{\mathcal{F}}(P)$.

- (I.3) For each $f \in \text{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, $f \circ \delta_P(g) = \delta_Q(\pi f(g)) \circ f$.

Recently, in 2013, A. Chermak [Che13] proved the existence and uniqueness of centric linking systems associated to saturated fusion systems (see also [Oli13]).

Theorem 2.3. ([Che13],[Oli13]) *Let \mathcal{F} be a saturated fusion system on a finite p -group S . Then there exists a centric linking system \mathcal{L} associated to \mathcal{F} . Moreover, \mathcal{L} is uniquely determined by \mathcal{F} up to isomorphism.*

Definition 2.4. The *classifying space $B\mathcal{F}$* of a saturated fusion system (S, \mathcal{F}) is the Bousfield-Kan p -completion of the nerve of the associated centric linking system $|\mathcal{L}|_p^\wedge$. We denote by $\Theta: BS \rightarrow B\mathcal{F}$ the map induced by the distinguished monomorphism δ_S .

Given a map $f: B\mathcal{F} \rightarrow X$ and $P \leq S$, we denote by $f|_{BP}$ the composite $f \circ \Theta \circ Bi$ where $i: P \leq S$.

We describe some results concerning the homotopy type of the classifying space $B\mathcal{F}$ and mapping spaces $\text{map}(BP, B\mathcal{F})$ which will be used in the rest of the paper.

Proposition 2.5 ([BLO03b],[BCG⁺07],[CL09]). *For any satured fusion system (S, \mathcal{F}) , $B\mathcal{F}$ is a p -complete space and $\pi_i(B\mathcal{F})$ are finite p -groups for all $i \geq 1$. The fundamental group $\pi_1(B\mathcal{F}) \cong S/\mathcal{O}_{\mathcal{F}}^p(S)$, where $\mathcal{O}_{\mathcal{F}}^p(S) := \langle [Q, \mathcal{O}^p(\text{Aut}_{\mathcal{F}}(Q))] \mid Q \leq S \rangle$.*

Proof. The description of the fundamental group $\pi_1(B\mathcal{F})$ is given in [BCG⁺07, Theorem B]. The fact that $B\mathcal{F}$ is p -complete follows from [BK72, Proposition I.5.2] since the nerve of the associated linking system $|\mathcal{L}|$ is p -good by [BLO03b, Proposition 1.12]. Finally, $\pi_i(B\mathcal{F})$ are finite p -groups for all $i \geq 1$ by [CL09, Lemma 7.6]. \square

Proposition 2.6. [BLO03b, Theorem 6.3] *Let (S, \mathcal{F}) be a saturated fusion system. If P is a finite p -group and $\rho: P \rightarrow S$ a group homomorphism such that $\rho(P)$ is fully \mathcal{F} -centralized, there is a saturated fusion system $(C_S(\rho(P)), C_{\mathcal{F}}(\rho(P)))$ and a homotopy equivalence $BC_{\mathcal{F}}(\rho(P)) \xrightarrow{\cong} \text{map}(BP, B\mathcal{F})_P$. In particular, the evaluation map $\text{map}(BP, B\mathcal{F})_c \rightarrow B\mathcal{F}$ is a homotopy equivalence.*

Moreover, any finite covering of the classifying space of a saturated fusion system is again the classifying space of a saturated fusion system.

Theorem 2.7 ([BCG⁺07, Theorem A]). *Let (S, \mathcal{F}) be a saturated fusion system and \mathcal{L} be the associated linking system. Then there is a normal subgroup $H \triangleleft \pi_1|\mathcal{L}|$ which is minimal among all those whose quotient is finite and p -solvable. Any covering space of the geometric realization $|\mathcal{L}|$ whose fundamental group contains H is homotopy equivalent to $|\mathcal{L}'|$ for some linking system \mathcal{L}' associated to a saturated fusion system (S', \mathcal{F}') , where $S' \leq S$ and $\mathcal{F}' \leq \mathcal{F}$.*

In this work it is important to understand the homotopy properties of mapping spaces between classifying spaces. This question was already studied in [BLO03b]. One of the standard techniques used when studying maps between p -completed classifying spaces of finite groups is to replace them by the p -completion of a homotopy colimit of classifying spaces of subgroups.

Definition 2.8. Let \mathcal{F} be a saturated fusion system over a finite p -group S . The *orbit category* of \mathcal{F} is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S , and whose morphisms are defined by

$$\text{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \text{Rep}_{\mathcal{F}}(P, Q) := \text{Inn}(Q) / \text{Hom}_{\mathcal{F}}(P, Q).$$

We let $\mathcal{O}^c(\mathcal{F})$ denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the \mathcal{F} -centric subgroups of S . If \mathcal{L} is a centric linking system associated to \mathcal{F} , then $\tilde{\pi}$ denotes the composite functor

$$\tilde{\pi}: \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c \twoheadrightarrow \mathcal{O}^c(\mathcal{F}).$$

The homotopy type of the nerve of a centric linking system can be described as a homotopy colimit over the orbit category.

Proposition 2.9 ([BLO03b, Proposition 2.2]). *Fix a saturated fusion system \mathcal{F} over a finite p -group S and an associated centric linking system \mathcal{L} , and let $\tilde{\pi} \rightarrow \mathcal{O}^c(\mathcal{F})$ be the projection functor. Let $\tilde{B}: \mathcal{O}^c(\mathcal{F}) \rightarrow \text{Top}$ be the left homotopy Kan extension over $\tilde{\pi}$ of the constant functor $\mathcal{L} \xrightarrow{*} \text{Top}$. Then \tilde{B} is a lift of the classifying space functor $P \mapsto BP$ to the category of topological spaces, and*

$$|\mathcal{L}| \simeq \text{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}).$$

Given fusion systems \mathcal{F} and \mathcal{F}' on S and S' respectively, a homomorphism $\psi: S \rightarrow S'$ is called *fusion preserving* if for every $\varphi \in \mathcal{F}(P, Q)$ there exists some $\varphi' \in \mathcal{F}'(\psi(P), \psi(Q))$ such that $\psi \circ \varphi = \varphi' \circ \psi$.

Theorem 2.10 ([CL09, Theorem 1.3]). *Let (S, \mathcal{F}) and (S', \mathcal{F}') be saturated fusion systems. Suppose that $\rho: S \rightarrow S'$ is a fusion preserving homomorphism. Then there exists some $m \geq 0$ and a map $\tilde{f}: B\mathcal{F} \rightarrow B(\mathcal{F}' \wr \Sigma_{p^m})$ such that the diagram below commutes up to homotopy*

$$\begin{array}{ccccc} BS & \xrightarrow{\Theta} & B\mathcal{F} & & \\ B\rho \downarrow & & \searrow \tilde{f} & & \\ BS' & \xrightarrow{\Theta'} & B\mathcal{F}' & \xrightarrow{\Delta_p^\wedge} & B(\mathcal{F}' \wr \Sigma_{p^m}). \end{array}$$

where $\mathcal{F}' \wr \Sigma_n$ is the saturated fusion system whose classifying space is $((B\mathcal{F}')_{h\Sigma_n}^n)_p^\wedge$.

3 The kernel of a map from a classifying space

Given a connected space A , we say that a space X is A -null if the evaluation map $ev: \text{map}(A, X) \rightarrow X$ is a weak equivalence (see [DF96]). If X is connected, X is A -null iff $\text{map}_*(A, X)$ is weakly contractible. There is a nullification functor $P_A: \text{Spaces} \rightarrow \text{Spaces}$ with a natural transformation $\eta_X: X \rightarrow P_A(X)$ which is initial with respect to maps into A -null spaces. We say that X is A -acyclic if $P_AX \simeq *$.

The kernel of a map $f: BG_p^\wedge \rightarrow Z$, where G is a compact Lie group and Z is a connected p -complete $\Sigma B\mathbb{Z}/p$ -null space, is defined by D. Notbohm in [Not94]. We adapt his definition to our context.

Definition 3.1. Let (S, \mathcal{F}) be a saturated fusion system and let Z be a connected p -complete $\Sigma B\mathbb{Z}/p$ -null space. If $f: B\mathcal{F} \rightarrow Z$ is a pointed map, we define the kernel of f

$$\ker(f) := \{g \in S \mid f|_{B\langle g \rangle} \simeq *\}.$$

Remark 3.2. By Proposition 2.6, we have $\text{map}(B\mathbb{Z}/p, B\mathcal{F})_c \simeq B\mathcal{F}$. It follows then by looping that $\Omega B\mathcal{F}$ is $B\mathbb{Z}/p$ -null, or equivalently, that $B\mathcal{F}$ is $\Sigma B\mathbb{Z}/p$ -null ([DF96, 3.A.1]).

Remark 3.3. If X is a $B\mathbb{Z}/p$ -null space, then X is BP -null for any finite p -group P . There are weak equivalences $\text{map}_*(BP, X) \simeq \text{map}_*(P_{B\mathbb{Z}/p}(BP), X) \simeq *$, where the last equivalence follows from Lemma 6.13 in [Dwy96] which states that BP is $B\mathbb{Z}/p$ -acyclic. A direct proof can be obtained by induction using the central extension of a p -group and Zabrodsky's Lemma [Dwy96, Proposition 3.4].

We will show that $\ker(f)$ is a subgroup of S with some important properties.

Definition 3.4. Let \mathcal{F} be a fusion system over a finite p -group S . Then a subgroup $K \leq S$ is *strongly \mathcal{F} -closed* if for all $P \leq K$ and all morphism $\varphi: P \rightarrow S$ in \mathcal{F} we have $\varphi(P) \leq K$.

If G is a finite group and $S \in \text{Syl}_p(G)$, $K \leq S$ is strongly $\mathcal{F}_S(G)$ -closed if and only if K is *strongly closed in G* , i.e., if for all $k \in K$ and $g \in G$ such that $c_g(s) \in S$, then $c_g(s) \in K$.

Since the intersection of strongly \mathcal{F} -closed subgroups is again strongly \mathcal{F} -closed, given a finite p -group P , we define $Cl_{\mathcal{F}}(P)$ to be the smallest strongly \mathcal{F} -closed subgroup of S that contains $f(P)$ for all $f \in \text{Hom}(P, S)$.

Proposition 3.5. Let $f: B\mathcal{F} \rightarrow Z$ be a pointed map as in Definition 3.1. The kernel $\ker(f)$ is a strongly \mathcal{F} -closed subgroup of S .

Proof. Since $\langle x \rangle = \langle x^{-1} \rangle$, in order to show that $\ker(f)$ is a subgroup of S it is sufficient to prove that if $x, y \in \ker(f)$, then $xy \in \ker(f)$. The composite $B\langle x, y \rangle \rightarrow BS \rightarrow B\mathcal{F} \rightarrow Z$ is null-homotopic by [Not94, Proposition 2.4]. Since $\langle xy \rangle \hookrightarrow \langle x, y \rangle$, $f|_{B\langle xy \rangle} \simeq *$.

Let $P \leq \ker(f)$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$. We have the following homotopy commutative diagram

$$\begin{array}{ccc} BP & \xrightarrow{B\delta_P} & B\mathcal{F} \\ B\varphi \downarrow \simeq & \nearrow B\delta_{\varphi(P)} & \xrightarrow{f} Z \\ B\varphi(P) & & \end{array}$$

which shows that $f|_{BP}$ is null-homotopic if and only if $f|_{B\varphi(P)}$ is null-homotopic. \square

W. Dwyer shows in [Dwy96, Theorem 5.1] that if we have a finite group G , a map $f: BG_p^\wedge \rightarrow Z$ as in Definition 3.1 is null-homotopic if and only if $\ker(f) = S$. This statement is also true for classifying spaces of saturated fusion systems.

Theorem 3.6. *Let (S, \mathcal{F}) be a saturated fusion system. Let Z be a connected p -complete $\Sigma B\mathbb{Z}/p$ -null space. Then a map $f: B\mathcal{F} \rightarrow Z$ is null-homotopic if and only if $\ker(f) = S$.*

Proof. If $f \simeq *$, then $f|_{BS} \simeq *$ and therefore $\ker(f) = S$. Now assume that $f|_{BS} \simeq *$, we will show that $f \simeq *$.

Step 1: Assume that $\pi_1(Z)$ is abelian. By Proposition 2.9, $B\mathcal{F} \simeq (\text{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B}P)_p^\wedge$, where $\tilde{B}P \simeq BP$ for $P \in \mathcal{F}^c$. Since any map $BP \rightarrow B\mathcal{F}$ factors through $\Theta: BS \rightarrow B\mathcal{F}$ by [BLO03b, Theorem 4.4], $f|_{BP} \simeq *$ for all $P \in \mathcal{F}^c$. Therefore we have two maps

$$\text{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}P) \rightarrow (\text{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}P))_p^\wedge \xrightarrow[\simeq]{f} Z,$$

such that both are nullhomotopic when restricted to BP for all $P \in \mathcal{F}^c$.

The obstructions for these maps to be homotopic are in $\lim_{\mathcal{O}^c(\mathcal{F})}^i \pi_i(\text{map}(BP, Z)_c)$, for $i \geq 1$ (see [Woj87]). Since a $B\mathbb{Z}/p$ -null space is BQ -null for any finite p -group Q (Remark 3.3), Z is ΣBP -null and hence $\text{map}_*(BP, Z)$ is homotopically discrete, therefore $\text{map}_*(BP, Z)_c \simeq *$ and, from the fibration $\text{map}_*(BP, Z)_c \rightarrow \text{map}(BP, Z)_c \rightarrow Z$, we obtain $\text{map}(BP, Z)_c \simeq Z$.

Given a morphism $\varphi: P \rightarrow Q$ in \mathcal{F}^c , $B\varphi$ is a pointed map and induces a commutative diagram

$$\begin{array}{ccc} \text{map}(BQ, Z)_c & \xrightarrow{B\varphi^*} & \text{map}(BP, Z)_c \\ \simeq \downarrow \text{ev} & & \text{ev} \downarrow \simeq \\ Z & \xrightarrow{id} & Z \end{array}$$

which shows that the obstructions are in $\lim_{\mathcal{O}^c(\mathcal{F})}^i \pi_i Z$, where $\pi_* Z$ is a constant functor in $\mathcal{O}^c(\mathcal{F})$. Since Z is p -complete and $\pi_1(Z)$ is abelian, the constant functors $\pi_i(Z)$ all take values in $\mathbb{Z}_{(p)} - \text{Mod}$.

Let $F := \pi_* Z$ be the constant functor. Fix P in $\mathcal{O}^c(\mathcal{F})^{\text{op}}$ and consider the atomic functors $F_P: \mathcal{O}^c(\mathcal{F})^{\text{op}} \rightarrow \mathbb{Z}_{(p)} - \text{Mod}$

$$F_P(Q) := \begin{cases} \pi_* Z & , \text{ if } Q = P, \\ 0 & , \text{ if } Q \neq P. \end{cases}$$

and $\tilde{F}_P(Q) := F(Q)/F_P(Q)$. Filtering F as a series of extensions of functors F_P , one for each object P , and using the long exact sequence for higher limits (see proof of Theorem 1.10 of [JMO95]), if $\lim^i F_P = 0$ for all P and $i > 0$, then $\lim^i F = 0$ for $i > 0$. By [JMO92, Proposition 6.1 (i),(ii)], if $p \nmid |\text{Out}_{\mathcal{F}}(P)|$, then

$$\lim^i F_P = \begin{cases} (\pi_* Z)^{\text{Out}_{\mathcal{F}}(P)} = \pi_* Z & , \text{ if } i = 0, \\ 0 & , \text{ if } i > 0. \end{cases}$$

and if $p \mid |\text{Out}_{\mathcal{F}}(P)|$, then $\lim^i F_P = 0$ for $i \geq 0$. Therefore $\lim_{O^*(\mathcal{F})}^i \pi_i Z = 0$ for $i > 0$, and finally $f \simeq *$.

Step 2: We will prove that a map $f: B\mathcal{F} \rightarrow BG$ is null-homotopic if $f|_{BS}: BS \rightarrow BG$ is null-homotopic where G is a discrete group.

We will apply Zabrodsky's lemma [Dwy96, Proposition 3.5] to the map $f|_{BS}$ and the fibre sequence $F \rightarrow BS \rightarrow B\mathcal{F}$ where F is connected since $\pi_1(\Theta)$ is an epimorphism (Lemma 2.5). Note that $\text{map}_*(F, BG)$ is homotopically discrete, then Zabrodsky's lemma shows that there is a homotopy equivalence $\text{map}(B\mathcal{F}, BG) \simeq \text{map}(BS, BG)_{[c]}$ where that last mapping space corresponds to the components which are null-homotopic when restricted to F . The bijection between components implies that $f \simeq *$.

Finally, we can prove the theorem by looking at the universal cover \tilde{Z} of Z . Let $p: Z \rightarrow B\pi_1(Z)$. The composite $p \circ f$ is null-homotopic by Step 2. Therefore there is a lift $\tilde{f}: B\mathcal{F} \rightarrow \tilde{Z}$ to the universal cover of Z . In order to apply Step 1, we need to check that $\tilde{f}|_{BS}: BS \rightarrow \tilde{Z}$ is null-homotopic. Since both Z and $B\pi_1(Z)$ are $\Sigma B\mathbb{Z}/p$ -null, applying mapping spaces from BS to the universal cover fibration shows that $\text{map}(BS, \tilde{Z})_{\{c\}} \simeq \tilde{Z}$ where $\{c\}$ is the set of maps which are nullhomotopic when postcomposed with $\tilde{Z} \rightarrow Z$. Note that $[\tilde{f}|_{BS}] \in \{c\}$. Since \tilde{Z} is connected, the set $\{c\}$ only consists on the constant map and then $\tilde{f}|_{BS}$ is nullhomotopic. \square

Given a strongly \mathcal{F} -closed subgroup $K \leq S$ in a saturated fusion system \mathcal{F} , there exists a map between classifying spaces $f: B\mathcal{F} \rightarrow BG_p^\wedge$ for a finite group G such that $\ker(f) = K$.

Proposition 3.7. *Let (S, \mathcal{F}) be a saturated fusion system and K be a strongly \mathcal{F} -closed subgroup and \mathcal{L} be the associated centric linking system. Let ρ be the composite $S \xrightarrow{\pi} S/K \xrightarrow{\text{reg}} \Sigma_{|S/K|}$, where π is the quotient homomorphism and reg is the regular representation of S/K . Then there is a non-negative integer $m \geq 0$ and a map $f: |\mathcal{L}| \rightarrow B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ such that the following diagram*

$$\begin{array}{ccc} BS & \xrightarrow{(\Delta B\rho)_p^\wedge} & (B(\Sigma_{|S/K|})^{p^m})_p^\wedge \\ \downarrow \iota_S & & \downarrow \Delta_p^\wedge \\ |\mathcal{L}| & \xrightarrow{f} & B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge \end{array}$$

is commutative up to homotopy. Moreover, $\ker(f_p^\wedge) = K$.

Proof. Let $n = |S/K|$. According to [CL09, Theorem 1.2], if ρ is fusion invariant then there is a non-negative integer $m \geq 0$ and a map $f: |\mathcal{L}| \rightarrow B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ such that $f|_{BS}$ is homotopic

to the composite $BS \xrightarrow{(\Delta B\rho)_p^\wedge} (B(\Sigma_n)^{p^m})_p^\wedge \xrightarrow{\Delta_p^\wedge} B(\Sigma_n \wr \Sigma_{p^m})_p^\wedge$. Therefore, it is sufficient to show that ρ is fusion invariant: for all $P \leq S$ and $\varphi: P \rightarrow S$ in \mathcal{F} there is $\omega \in \Sigma_n$ such that $\rho|_{\varphi(P)} \circ \varphi = c_\omega \circ \rho|_P$.

The homomorphisms $\rho|_P$ and $c_\omega \rho|_P$ equip S/K with a structure of a P -set which are isomorphic. Hence, to prove the above equality, we only need to show that $(S/K, \leq)$ and $(S/K, \leq_\varphi)$ are equivalent as P -sets.

Note that for any $\varphi: P \rightarrow S \in \mathcal{F}$,

$$(S/K, \leq_\varphi) \cong \text{Iso}^*(\varphi) \text{Res}_{\varphi(P)}^S(S/K) \cong \text{Iso}^*(\varphi) \text{Res}_{\varphi(P)}^S \text{Ind}_K^S(*).$$

Applying the Mackey formula to $\text{Res}_{\varphi(P)}^S \text{Ind}_K^S$, we get

$$\begin{aligned} (S/K, \leq_\varphi) &\cong \coprod_{[x] \in \varphi(P) \backslash S/K} \text{Iso}^*(\varphi) \text{Ind}_{\varphi(P) \cap K^x}^{\varphi(P)} \text{Iso}^*(c_x) \text{Res}_{(\varphi(P) \cap K)^x}^K(*) \\ &= \coprod_{[x] \in \varphi(P) \backslash S/K} \text{Ind}_{\varphi^{-1}(\varphi(P) \cap K)}^P \text{Iso}^*(\varphi) \text{Iso}^*(c_x) \text{Res}_{(\varphi(P) \cap K)^x}^K(*). \end{aligned}$$

where the second equality comes from the commutativity of isogation and induction and, where $K^x = K$ because K is strongly \mathcal{F} -closed and $c_x: K \rightarrow S$ is in \mathcal{F} , hence $c_x(K) = K^x \leq K$ and since c_x is an isomorphism, $K^x = K$.

Next we prove that $\varphi^{-1}(\varphi(P) \cap K) = P \cap K$. On the one hand, $\varphi^{-1}(\varphi(P) \cap K) \leq P \cap K$ because $\varphi^{-1}|_{\varphi(P) \cap K}: \varphi(P) \cap K \rightarrow S$ is in \mathcal{F} , $\varphi(P) \cap K \leq K$, K is strongly \mathcal{F} -closed and $\varphi^{-1}(\varphi(P) \cap K) \leq P$. The equality will follow if $|\varphi^{-1}(\varphi(P) \cap K)| = |P \cap K|$. Since φ is an isomorphism, it is enough to check $|\varphi(P) \cap K| = |\varphi(P \cap K)|$. We already know that $|\varphi(P) \cap K| = |\varphi^{-1}(\varphi(P) \cap K)| \leq |P \cap K| = |\varphi(P \cap K)|$. Since $\varphi|_{P \cap K}: P \cap K \rightarrow S$ is in \mathcal{F} , $P \cap K \leq K$ and K is strongly \mathcal{F} -closed, $\varphi(P \cap K) \leq K$ but also $\varphi(P \cap K) \leq \varphi(P)$, hence $\varphi(P \cap K) \leq \varphi(P) \cap K$ and therefore $|\varphi(P \cap K)| \leq |\varphi(P) \cap K|$.

Therefore in the above formula, since $\text{Iso}^*(\varphi) \text{Iso}^*(c_x) \text{Res}_{(\varphi(P) \cap K)^x}^K(*) = *$ as $(P \cap K)$ -set and $\varphi^{-1}(\varphi(P) \cap K) = P \cap K$, we get for all $\varphi: P \rightarrow S \in \mathcal{F}$

$$(S/K, \leq_\varphi) \cong \coprod_{[x] \in \varphi(P) \backslash S/K} \text{Ind}_{P \cap K}^P(*) \cong \coprod_{l_\varphi} P/P \cap K,$$

where $l_\varphi = |\varphi(P) \backslash S/K| = |S/\varphi(P) \cdot K| = |S|/|\varphi(P) \cdot K|$ since $K \triangleleft S$. In particular, if $\varphi = id_P$, then

$$(S/K, \leq) \cong \coprod_{[x] \in P \backslash S/K} \text{Ind}_{P \cap K}^P(*) \cong \coprod_l P/P \cap K,$$

where $l = |S|/|P \cdot K|$.

Therefore, ρ will be fusion invariant if $l = l_\varphi$. It is enough to show $|\varphi(P) \cap K| = |P \cap K|$ and, in fact, this equality has been already proved in a paragraph above.

Finally, $f|_{BS}$ is the induced map on classifying spaces of the homomorphism

$$S \xrightarrow{\pi} S/K \xrightarrow{reg} \Sigma_{|S/K|} \longrightarrow \Sigma_{|S/K|} \wr \Sigma_{p^m}$$

whose kernel is K by construction. □

Question: Given a saturated fusion system (S, \mathcal{F}) and a map $f: B\mathcal{F} \rightarrow Z$ where Z is a connected $\Sigma B\mathbb{Z}/p$ -null p -complete space. Does f factors, up to homotopy, through $\tilde{f}: B\mathcal{F}' \rightarrow Z$ with trivial kernel, where \mathcal{F}' is a saturated fusion system?

Under some hypothesis we can give a positive answer to the previous question.

Proposition 3.8. *Let (S, \mathcal{F}) be a saturated fusion system with associated linking system \mathcal{L} . Let K be the kernel of $f: B\mathcal{F} \rightarrow Z$, where Z is $\Sigma B\mathbb{Z}/p$ -null p -complete space.*

If K is normal in \mathcal{F} , then there exist a saturated fusion system $(S/K, \mathcal{F}/K)$ with associated linking system \mathcal{L}/K and a map $pr: |\mathcal{L}| \rightarrow |\mathcal{L}/K|$, whose homotopy fibre is BK , such that f factors via $\tilde{f}: B(\mathcal{F}/K) \rightarrow Z$ with trivial kernel.

Proof. In [OV07, Section 2], the authors prove that if K is normal in \mathcal{F} , then there is a saturated fusion system $(S/K, \mathcal{F}/K)$ with linking system \mathcal{L}/K and a map $pr: |\mathcal{L}| \rightarrow |\mathcal{L}/K|$ whose homotopy fibre is BK .

Since Z is p -complete, one can consider the composite $g: |\mathcal{L}| \rightarrow B\mathcal{F} \rightarrow Z$ such that $g_p^\wedge \simeq f$. By assumption Z is $\Sigma B\mathbb{Z}/p$ -null, and also BK is $B\mathbb{Z}/p$ -acyclic, then we get a unique factorization, up to homotopy, $\tilde{g}: |\mathcal{L}/K| \rightarrow Z$ of g by applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) to the homotopy fibre sequence $BK \rightarrow |\mathcal{L}| \rightarrow |\mathcal{L}/K|$ and the map g . Take $\tilde{f} = \tilde{g}_p^\wedge$. The same argument applied to the fibration $BK \rightarrow BS \xrightarrow{\pi} B(S/K)$ and $g|_{BS}$ shows that $\tilde{f}|_{B(S/K)} \circ B\pi \simeq f|_{BS}$, by uniqueness.

Let $[x] \in \ker(\tilde{f}) \leq S/K$, then $\tilde{f}|_{B([x])} \simeq *$ implies that $f|_{B\langle x \rangle} \simeq *$ and finally $x \in K$. \square

Corollary 3.9. *Let (S, \mathcal{F}) be a saturated fusion system with associated linking system \mathcal{L} . Let K be the kernel of $f: B\mathcal{F} \rightarrow Z$, where Z is a connected $\Sigma B\mathbb{Z}/p$ -null p -complete space.*

If K is abelian or S is resistant, then there exist a saturated fusion system $(S/K, \mathcal{F}/K)$ with associated linking system \mathcal{L}/K and a map $pr: |\mathcal{L}| \rightarrow |\mathcal{L}/K|$ whose homotopy fibre is BK and a homotopy factorization $\tilde{f}: B\mathcal{F}/K \rightarrow Z$ with trivial kernel.

Proof. Recall that K is strongly \mathcal{F} -closed by Proposition 3.5. If K is abelian and strongly \mathcal{F} -closed, then it is normal in \mathcal{F} by [Cra10, Proposition 3.14]. If S is resistant, each strongly \mathcal{F} -closed subgroup is also normal in \mathcal{F} . Finally, apply Proposition 3.8. \square

4 Homotopy properties of BP -cellular approximations

In this context, it is natural to study homotopy properties of $CW_{BP}X$ which are inherited by those of X . We will show in this section that $CW_{BP}(B\mathcal{F})$ is a p -good nilpotent space whose fundamental group is a finite p -group.

Proposition 4.1. *Let (S, \mathcal{F}) be a saturated fusion system. Then both $B\mathcal{F}$ and $CW_{BP}(B\mathcal{F})$ are nilpotent. Furthermore, there exists a homotopy fibre sequence*

$$CW_{BP}(B\mathcal{F}) \rightarrow CW_{BP}(B\mathcal{F})_p^\wedge \rightarrow (CW_{BP}(B\mathcal{F})_p^\wedge)_{\mathbb{Q}}.$$

Proof. The classifying space $B\mathcal{F}$ is p -complete and $\pi_i(B\mathcal{F})$ are finite p -groups by Lemma 2.5, hence $B\mathcal{F}$ is nilpotent according to [BK72, Proposition VII.4.3(ii)]. Then $CW_{BP}(B\mathcal{F})$ is also nilpotent [CF15, Corollary 3.2].

Moreover, it follows from [CF15, Lemma 2.8] that $R_\infty CW_{BP}(B\mathcal{F}) \simeq *$ for $R = \mathbb{Q}$ and $R = \mathbb{F}_q$, $q \neq p$, since $\tilde{H}_*(BP; R) = 0$. Finally, the Sullivan's arithmetic square applied to the nilpotent space $CW_{BP}(B\mathcal{F})$ gives the desired result. \square

The R -completion functor is a homological localization functor when we restrict to R -good spaces (see [BK72, p. 205]). By [BK72, Proposition VII.5.1], if the fundamental group of a pointed space X is finite, then X is p -good for any prime p . Therefore, in order to show that $CW_{BP}(B\mathcal{F})$ is a p -good space, we will prove that its fundamental group is a finite p -group.

Proposition 4.2. *Let (S, \mathcal{F}) be a saturated fusion system and let P be a finite p -group. Then $\pi_1 CW_{BP}(B\mathcal{F})$ is a finite p -group. Therefore, $CW_{BP}(B\mathcal{F})$ is a p -good space.*

Proof. The proof will be divided in several steps. Let C be the homotopy cofibre of the evaluation map $ev: \bigvee_{[BP, B\mathcal{F}]_*} BP \rightarrow B\mathcal{F}$.

Step 1: Assume C is simply connected. The Chacholski's homotopy fibre sequence $CW_{BP}(B\mathcal{F}) \rightarrow B\mathcal{F} \rightarrow P_{\Sigma BP}C$ induces a long exact sequence of homotopy groups

$$\dots \rightarrow \pi_2(P_{\Sigma BP}C) \rightarrow \pi_1 CW_{BP}(B\mathcal{F}) \rightarrow \pi_1(B\mathcal{F}) \rightarrow \dots$$

where $\pi_1(B\mathcal{F})$ is a finite p -group by Lemma 2.5. Therefore, we are reduced to prove that $\pi_2(P_{\Sigma BP}C)$ is a finite p -group.

Hurewicz's theorem shows that $H_2(P_{\Sigma BP}C; \mathbb{Z}) \cong \pi_2(P_{\Sigma BP}C)$, since $P_{\Sigma BP}C$ is simply connected ([Bou94, 2.9]). Moreover, since ΣBP is 1-connected, we obtain an epimorphism $H_2(C; \mathbb{Z}) \rightarrow H_2(P_{\Sigma BP}C; \mathbb{Z})$ by [CGR15, Proposition 3.2]. Then it is enough to prove that $H_2(C; \mathbb{Z})$ is a finite p -group.

The cofibration sequence $\bigvee_{[BP, B\mathcal{F}]_*} BP \rightarrow B\mathcal{F} \rightarrow C$ induces a long exact sequence of homology groups

$$\dots \rightarrow H_2(B\mathcal{F}; \mathbb{Z}) \xrightarrow{f_1} H_2(C; \mathbb{Z}) \xrightarrow{f_2} H_1(\bigvee_{[BP, B\mathcal{F}]_*} BP; \mathbb{Z}) \rightarrow \dots$$

which allows to describe $H_2(C; \mathbb{Z})$ by a short exact sequence

$$0 \rightarrow \ker f_2 \rightarrow H_2(C; \mathbb{Z}) \rightarrow \operatorname{Im} f_2 \rightarrow 0.$$

By exactness, $\ker f_2 = \operatorname{Im} f_1$ is a quotient of $H_2(B\mathcal{F}; \mathbb{Z})$, where $H_2(B\mathcal{F}; \mathbb{Z})$ is a finite p -group since $H^*(B\mathcal{F}; \mathbb{Z}) \hookrightarrow H^*(BS; \mathbb{Z})$ by [BLO03b, Theorem B]. Hence $\ker f_2$ is a finite p -group.

Now note that

$$\operatorname{Im} f_2 \subset H_1\left(\bigvee_{[BP, B\mathcal{F}]_*} BP; \mathbb{Z}\right) \cong \pi_1\left(\bigvee_{[BP, B\mathcal{F}]_*} BP\right)_{ab} \cong \bigoplus_{[BP, B\mathcal{F}]_*} P_{ab},$$

where $[BP, B\mathcal{F}]_*$ is a finite set because there is an epimorphism of sets $[BP, BS]_* \twoheadrightarrow [BP, B\mathcal{F}]_*$ by [BLO03b, Theorem 4.4], and $[BP, BS]_* \cong \operatorname{Hom}(P, S)$ is finite. Finally, $H_2(C; \mathbb{Z})$ is a finite p -group.

Step 2: We will show that there exists a saturated fusion system (S', \mathcal{F}') and a BP -equivalence $f: B\mathcal{F}' \rightarrow B\mathcal{F}$ such that the homotopy cofibre of $ev: \bigvee_{[BP, B\mathcal{F}']_*} BP \rightarrow B\mathcal{F}'$ is 1-connected.

Let N be the normal subgroup of $\pi_1 B\mathcal{F}$ generated by the image of homomorphisms $P \rightarrow \pi_1 B\mathcal{F}$ such that the induced pointed map $BP \rightarrow B\pi_1 B\mathcal{F}$ lifts to $B\mathcal{F}$. Let X be the

pullback of the universal cover $\widetilde{B\mathcal{F}} \rightarrow B\mathcal{F} \rightarrow B\pi_1 B\mathcal{F}$ along $BN \rightarrow B\pi_1 B\mathcal{F}$. Then we have a homotopy commutative diagram

$$\begin{array}{ccccc} \widetilde{B\mathcal{F}} & \xlongequal{\quad} & \widetilde{B\mathcal{F}} & & \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{f} & B\mathcal{F} & \longrightarrow & B(\pi_1 B\mathcal{F}/N) \\ \downarrow & & \downarrow & & \parallel \\ BN & \rightarrow & B\pi_1 B\mathcal{F} & \rightarrow & B(\pi_1 B\mathcal{F}/N) \end{array}$$

where the vertical arrows are homotopy fibrations and the horizontal arrows are principal homotopy fibrations. By Theorem 2.7, there is a saturated fusion system (S', \mathcal{F}') such that $S' \leq S$, $\mathcal{F}' \leq \mathcal{F}$ and $X \simeq B\mathcal{F}'$. Furthermore, $f: B\mathcal{F}' \simeq X \rightarrow B\mathcal{F}$ is a BP -equivalence by Proposition 4.3.

Let C be the homotopy cofibre of $ev: \bigvee_{[BP, B\mathcal{F}']} BP \rightarrow B\mathcal{F}'$. By Seifert-Van Kampen's theorem, C will be 1-connected if $\pi_1(ev): *_{[BP, B\mathcal{F}']} P \rightarrow \pi_1(B\mathcal{F}') \cong N$ is an epimorphism.

Since N is generated by the image of homomorphisms $f: P \rightarrow \pi_1 B\mathcal{F}$ such that the induced map $Bf: BP \rightarrow B\pi_1 B\mathcal{F}$ lifts to $\tilde{f}: BP \rightarrow B\mathcal{F}$, it is enough to show that for each of those f , $\text{Im}(f) \leq \text{Im}(\pi_1(ev))$.

By definition, there is a factorization $f: P \rightarrow N \leq \pi_1 B\mathcal{F}$ which lifts to $B\mathcal{F}$. There exists a unique $\beta: BP \rightarrow B\mathcal{F}'$ (up to homotopy) such that the diagram

$$\begin{array}{ccccc} BP & & \xrightarrow{\tilde{f}} & & B\mathcal{F} \\ & \searrow \beta & \downarrow & \searrow & \downarrow \\ & & B\mathcal{F}' & \longrightarrow & B\mathcal{F} \\ & \searrow f & \downarrow & & \downarrow \\ & & BN & \longrightarrow & B\pi_1 B\mathcal{F} \end{array}$$

is homotopy commutative. Therefore $\text{Im}(\pi_1(\beta)) = \text{Im}(f) \leq \text{Im}(\pi_1(ev))$.

Now we are ready to complete the proof. By Step 2 there exists a saturated fusion system (S', \mathcal{F}') and a BP -equivalence $f: B\mathcal{F}' \rightarrow B\mathcal{F}$ such that the homotopy cofibre of $ev: \bigvee_{[BP, B\mathcal{F}']} BP \rightarrow B\mathcal{F}'$ is 1-connected. Hence $CW_{BP}(B\mathcal{F}) \simeq CW_{BP}(B\mathcal{F}')$ and $\pi_1 CW_{BP}(B\mathcal{F}) \cong \pi_1 CW_{BP}(B\mathcal{F}')$ is a finite p -group by Step 1. \square

For the proof of Proposition 4.2 we need the next version of a result of N. Castellana, J.A. Crespo and J. Scherer.

Proposition 4.3 ([CCS07, Proposition 2.1]). *Let P be a finite p -group and let $F \twoheadrightarrow E \xrightarrow{\pi} BG$ be a fibration, where G is a discrete group. Let N be the normal subgroup of G generated by the image of homomorphisms $P \rightarrow G$ such that the induced pointed map $BP \rightarrow BG$ lifts to E . Then the pullback of the fibration along $BN \rightarrow BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{pr} & B(G/N) \\ \downarrow & & \downarrow \pi & \searrow pr' & \parallel \\ BN & \rightarrow & BG & \rightarrow & B(G/N) \end{array}$$

induces a BP -equivalence $f: E' \rightarrow E$ on the total space level.

Proof. We want to show that f induces a BP -equivalence. The top fibration in the diagram yields a fibration

$$\mathrm{map}_*(BP, E') \xrightarrow{f_*} \mathrm{map}_*(BP, E)_{\{c\}} \xrightarrow{pr_*} \mathrm{map}_*(BP, B(G/N))_{c'}.$$

Since the base space is homotopically discrete, we only need to check that all components of $\mathrm{map}_*(BP, E)$ are sent by pr_* to $\mathrm{map}_*(BP, B(G/N))_c$. Thus consider a pointed map $h: BP \rightarrow E$. The composite $pr \circ h$ is homotopy equivalent to a map induced by a group homomorphism $\alpha = \pi_1(pr \circ h): P \rightarrow G$ whose image is in N by construction. Therefore $pr \circ h \simeq pr' \circ \pi \circ h$ is null-homotopic. \square

Remark 4.4. In the original version the authors consider $P = B\mathbb{Z}/p^m$ for $m > 1$ and \bar{N} to be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \leq r$ such that the inclusion $f_g: B\langle g \rangle \rightarrow BG$ lifts to E , but this subgroup does not have the desired properties. Consider the fibration $B\mathbb{Z}/2 \xrightarrow{\iota} B\mathbb{Z}/4 \xrightarrow{pr} B\mathbb{Z}/2$, this fibration has no section and hence $\bar{N} = \{0\}$. Then $E' \simeq B\mathbb{Z}/2$ and $CW_{B\mathbb{Z}/4}(B\mathbb{Z}/2) \simeq B\mathbb{Z}/2 \neq B\mathbb{Z}/4 = CW_{B\mathbb{Z}/4}(B\mathbb{Z}/4)$, contradicting the proposition. However, $N \cong \mathbb{Z}/2 \cong \langle g \rangle$, since $f_g = pr$. Hence $E' \simeq B\mathbb{Z}/4$ and $f: E' \rightarrow B\mathbb{Z}/4$ is an equivalence, in particular it is a $B\mathbb{Z}/4$ -equivalence.

5 Cellular properties of the classifying space of a saturated fusion system

The goal of this section is to prove the main theorem of the paper. Given a finite p -group P , this result characterizes the property of being BP -cellular for classifying spaces of saturated fusion systems in terms of the fusion data.

Theorem 5.1. *Let (S, \mathcal{F}) be a saturated fusion system and let P be a finite p -group. Then $B\mathcal{F}$ is BP -cellular if and only if $S = Cl_{\mathcal{F}}(P)$.*

Corollary 5.2. *Let (S, \mathcal{F}) be a saturated fusion system and let P be a finite p -group.*

- (a) *The classifying space $B\mathcal{F}$ is BS -cellular.*
- (b) *Let (S, \mathcal{F}') be a saturated fusion system with $\mathcal{F} \subset \mathcal{F}'$. If $B\mathcal{F}$ is BP -cellular then $B\mathcal{F}'$ is also BP -cellular.*
- (c) *Let A be a pointed connected space. If $Cl_{\mathcal{F}}((\pi_1 A)_{ab}) = S$, then $B\mathcal{F}$ is A -cellular.*
- (d) *Let $\Omega_{p^m}(S)$ be the (normal) subgroup of S generated by its elements of order p^i , with $i \leq m$. Then $B\mathcal{F}$ is $B\mathbb{Z}/p^m$ -cellular if and only if $S = Cl_{\mathcal{F}}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $B\mathbb{Z}/p^m$ -cellular for all $m \geq m_0$.*

Proof. (a) Direct from Theorem 5.1 since $Cl_{\mathcal{F}}(S) = S$.

- (b) It follows from the the inclusions $Cl_{\mathcal{F}}(P) \leq Cl_{\mathcal{F}'}(P) \leq S$.
- (c) Notice that $SP^{\infty}A \simeq \prod_{i \geq 1} K(H_i(A; \mathbb{Z}), i)$ is A -cellular by [DF96, Corollary 4.A.2.1], so $B(\pi_1 A)_{ab} \simeq K(H_1(A; \mathbb{Z}), 1)$ is also A -cellular from [DF96, 2.D]. Then, $B\mathcal{F}$ is $B(\pi_1 A)_{ab}$ -cellular using (a).
- (d) We have $Cl_{\mathcal{F}}(\Omega_{p^m}(S)) \cong Cl_{\mathcal{F}}(\mathbb{Z}/p^m)$. Note that $B\Omega_{p^m}(S)$ is $B\mathbb{Z}/p^m$ -cellular and that there exists an $m_0 \geq 0$ such that S is generated by elements of order a power of p less than p^{m_0} .

□

The strategy of proof for Theorem 5.1 goes by analyzing the fibre sequence given in [Cha96, Theorem 20.5]

$$CW_{BP}(B\mathcal{F}) \xrightarrow{c} B\mathcal{F} \xrightarrow{r} P_{\Sigma BP}C,$$

where C is the homotopy cofibre of the evaluation map $ev: \bigvee_{[BP, B\mathcal{F}]_*} BP \rightarrow B\mathcal{F}$ and r is the composite $B\mathcal{F} \rightarrow C \rightarrow P_{\Sigma BP}C$.

The first goal is to compute the kernel of r_p^{\wedge} . In order to apply the theory of kernels, developed in Section 3, the target of the map needs to be a connected p -complete $\Sigma B\mathbb{Z}/p$ -null space.

Since $\pi_1 B\mathcal{F}$ is a finite p -group, the same holds for $P_{\Sigma BP}C$ [Bou94, 2.9], therefore Bousfield-Kan p -completion of the previous homotopy fibration is a homotopy fibre sequence ([BK72, II.5.1])

$$CW_{BP}(B\mathcal{F})_p^{\wedge} \xrightarrow{c_p^{\wedge}} B\mathcal{F} \xrightarrow{r_p^{\wedge}} (P_{\Sigma BP}C)_p^{\wedge}.$$

Lemma 5.3. *If X is a 1-connected space and P is a finite p -group, then $(P_{\Sigma BP}X)_p^{\wedge}$ is $\Sigma B\mathbb{Z}/p$ -null.*

Proof. We have the following weak homotopy equivalences

$$\mathrm{map}_*(\Sigma B\mathbb{Z}/p, (P_{\Sigma BP}X)_p^{\wedge}) \simeq \mathrm{map}_*(B\mathbb{Z}/p, \Omega(P_{\Sigma BP}X)_p^{\wedge}) \simeq \mathrm{map}_*(B\mathbb{Z}/p, (\Omega P_{\Sigma BP}X)_p^{\wedge})$$

where the last equivalence holds by [BK72, V.4.6 (ii)] since X is 1-connected (and so is $P_{\Sigma BP}X$ by [Bou94, 2.9]).

The commutation rules between nullification functors and loops in [DF96, 3.A.1] show that $\Omega P_{\Sigma BP}X \simeq P_{BP}(\Omega X)$. Finally, $\mathrm{map}_*(B\mathbb{Z}/p, (P_{BP}\Omega X)_p^{\wedge}) \simeq \mathrm{map}_*(B\mathbb{Z}/p, (P_{BP}\Omega X)) \simeq *$ where the first equivalence follows from Miller's theorem [Mil84, Thm 1.5] and the second from the fact that BP is $B\mathbb{Z}/p$ -acyclic ([Dwy96, Lemma 6.13]). □

Next we describe a criteria for detecting when a map from an A -cellular space is null-homotopic which will be useful later.

Proposition 5.4. *Let X and Y be pointed connected spaces. Assume that X is A -cellular and Y is ΣA -null. Then a pointed map $f: X \rightarrow Y$ is null-homotopic if and only if for any map $g: A \rightarrow X$ the composite $f \circ g$ is null-homotopic.*

Proof. If f is null-homotopic then for any map $g: A \rightarrow X$ the composite $f \circ g$ is null-homotopic. A

Assume that for any map $g: A \rightarrow X$, the composite $f \circ g$ is null-homotopic. Let C be the homotopy cofibre of $ev: \bigvee_{[A,X]_*} A \rightarrow X$. Since $f \circ ev \simeq *$ by assumption, there is a map $\tilde{f}: C \rightarrow Y$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} \bigvee_{[A,X]_*} A & & \\ \downarrow ev & \searrow \simeq * & \\ X & \xrightarrow{f} & Y \\ \downarrow & \searrow \tilde{f} & \\ C & & \end{array}$$

Then $f \simeq *$ if and only if $\tilde{f} \simeq *$.

Since X is A -cellular, $P_{\Sigma A} C \simeq *$. Finally $\tilde{f} \simeq *$ because $\text{map}_*(C, Y) \simeq \text{map}_*(P_{\Sigma A} C, Y) \simeq \text{map}_*(*, Y) \simeq *$, where the first equivalence follows from the fact that Y is ΣA -null. \square

A key step in the proof of Theorem 5.1 is the following computation of the kernel of the map r_p^\wedge .

Proposition 5.5. *Let (S, \mathcal{F}) be a saturated fusion system. Then $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$.*

Proof. We will show that $\ker(r_p^\wedge) \leq Cl_{\mathcal{F}}(P)$ and that $f(P) \leq \ker(r_p^\wedge)$ for all $f: P \rightarrow S$, then since $\ker(r_p^\wedge)$ is strongly \mathcal{F} -closed, the definition of $Cl_{\mathcal{F}}(P)$ implies the equality.

By universal properties of cellularization and p -completion, any map $BP \rightarrow B\mathcal{F}$ lifts to $CW_{BP}(B\mathcal{F})_p^\wedge$, then $f(P) \leq \ker(r_p^\wedge)$ for any homomorphism $f: P \rightarrow S$.

According to Proposition 3.7, there exist $m \geq 0$ and a pointed map between classifying spaces $k: B\mathcal{F} \rightarrow B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ such that $\ker(k) = Cl_{\mathcal{F}}(P)$. Let $\iota: B\ker(r_p^\wedge) \rightarrow B\mathcal{F}$ be the composite $B\ker(r_p^\wedge) \rightarrow BS \rightarrow B\mathcal{F}$. It is enough to show that $k \circ \iota$ is nullhomotopic.

There is a lift $\tilde{\iota}: B\ker(r_p^\wedge) \rightarrow CW_{BP}(B\mathcal{F})_p^\wedge$ such that following diagram is homotopy commutative

$$\begin{array}{ccccc} & & CW_{BP}(B\mathcal{F})_p^\wedge & & \\ & \nearrow \tilde{\iota} & \downarrow c & \searrow k \circ c & \\ B\ker(r_p^\wedge) & \xrightarrow{\iota} & B\mathcal{F} & \xrightarrow{k} & B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge \\ & \searrow * \simeq & \downarrow r_p^\wedge & & \\ & & (P_{\Sigma BP} C)_p^\wedge & & \end{array}$$

We will show that $k \circ c$ is nullhomotopic by applying Proposition 5.4. Recall that any map $f: BP \rightarrow B\mathcal{F}$ is homotopic to $\Theta \circ B\rho: BP \rightarrow BS \rightarrow B\mathcal{F}$ where $\rho \in \text{Hom}(P, S)$ (see [BLO03b, Theorem 4.4]). Then, the map k has the property that for any $f: BP \rightarrow B\mathcal{F}$, the composite $k \circ f$ is nullhomotopic.

In particular, if $c: CW_{BP}(B\mathcal{F}) \rightarrow B\mathcal{F}$ is the augmentation, for any $f: BP \rightarrow CW_{BP}(B\mathcal{F})$ we also have $(k \circ c) \circ f \simeq *$. Therefore $k \circ c \simeq *$ by Proposition 5.4 (since $B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ is $\Sigma B\mathbb{Z}/p$ -null and hence ΣBP -null by Remark 3.3). \square

An immediate consequence of Proposition 5.5 is one implication in Theorem 5.1, that is, if $B\mathcal{F}$ is BP -cellular then $S = Cl_{\mathcal{F}}(P)$. In order to prove the other implication, we will need a couple of technical results.

Lemma 5.6. *Let (S, \mathcal{F}) be a saturated fusion system. If $S = Cl_{\mathcal{F}}(P)$, then homotopy cofibre C of the evaluation $ev: \bigvee_{[BP, B\mathcal{F}]} BP \rightarrow B\mathcal{F}$ is 1-connected.*

Proof. From the cofibration sequence and Seifert-Van Kampen theorem, we have that $\pi_1 C \cong \pi_1 B\mathcal{F}/N$, where N is the minimal normal subgroup of $\pi_1 B\mathcal{F}$ containing $\text{Im}(\pi_1(ev))$. Given $f: BP \rightarrow B\mathcal{F}$, there is group homomorphism $g: P \rightarrow S$ such that $f \simeq \Theta \circ Bg$ (see [BLO03b, Theorem 4.4]). Let \bar{N} be normal subgroup of S generated by all $g(P)$, where $g \in \text{Hom}(P, S)$. Then the fundamental group can be described $\pi_1(C) \cong S/\bar{N}\mathcal{O}_{\mathcal{F}}^p(S)$.

First, $\bar{N}\mathcal{O}_{\mathcal{F}}^p(S)$ is strongly \mathcal{F} -closed by [DGPS11, Proposition A.9]. Moreover it contains $g(P)$ for all $g \in \text{Hom}(P, S)$. Therefore we have inclusions $Cl_{\mathcal{F}}(P) \leq \bar{N}\mathcal{O}_{\mathcal{F}}^p(S) \leq S$. Since we are assuming $S = Cl_{\mathcal{F}}(P)$, the previous inclusions are all equalities and C is then 1-connected. \square

Proposition 5.7. *Let Z be a $\Sigma^i BP$ -null space where P is a finite p -group and $i \geq 0$. If $\pi_1 Z$ is a finite p -group then Z_p^\wedge is also $\Sigma^i BP$ -null.*

Proof. Since $\pi_1 Z$ is a finite p -group, there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & Z\langle 1 \rangle & \longrightarrow & Z_p^\wedge\langle 1 \rangle \\ \parallel & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & Z_p^\wedge \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & B\pi_1(Z) & \xrightarrow{\simeq} & B\pi_1(Z_p^\wedge) \end{array}$$

From the proof of Proposition 3.1 in [DZ87], it is easy to see that $[\Sigma^i BP, Z_p^\wedge]_* \cong [\Sigma^i BP, Z]_* \cong *$. Then it is enough to check that $\text{map}_*(\Sigma^i BP, \Omega Z_p^\wedge) \simeq *$. Consider the homotopy fibre sequence $\Omega Z_p^\wedge \rightarrow F \rightarrow Z$. Since Z is $\Sigma^i BP$ -null, $\text{map}_*(\Sigma^i BP, \Omega Z_p^\wedge) \simeq \text{map}_*(\Sigma^i BP, F)$. This last mapping space is homotopy equivalent to $\text{map}_*(\Sigma^i BP, F_p^\wedge)$ by Miller's theorem [Mil84, Thm 1.5] since F is nilpotent. Finally $F_p^\wedge \simeq *$. \square

We are now ready to prove the main theorem of this paper.

Proof of Theorem 5.1. First assume that $B\mathcal{F}$ is BP -cellular. Then $P_{\Sigma BP}C$ is contractible and $r \simeq *$. This implies that $\ker(r_p^\wedge) = S$, but also $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$ by Proposition 5.5. Therefore $S = Cl_{\mathcal{F}}(P)$.

Now assume that $S = Cl_{\mathcal{F}}(P)$. Since C is 1-connected by Lemma 5.6 (and therefore $P_{\Sigma BP}C$ is so by [Bou94, 2.9]), consider the p -completed homotopy fibre sequence ([BK72, II.5.1])

$$CW_{BP}(B\mathcal{F})_p^\wedge \xrightarrow{c_p^\wedge} B\mathcal{F} \xrightarrow{r_p^\wedge} (P_{\Sigma BP}C)_p^\wedge.$$

We will show that $(P_{\Sigma BP}C)_p^\wedge$ is weakly contractible. For this we apply first the theory of kernels developed in Section 3 to r_p^\wedge (see Lemma 5.3 and Lemma 5.6). By assumption and Proposition 5.5, $\ker(r_p^\wedge) = S$ and therefore r_p^\wedge is null-homotopic by Theorem 3.6. Then there is a splitting $CW_{BP}(B\mathcal{F})_p^\wedge \simeq B\mathcal{F} \times \Omega(P_{\Sigma BP}C)_p^\wedge$.

Let $X = CW_{BP}(B\mathcal{F})_p^\wedge$ and $Y = \Omega(P_{\Sigma BP}C)_p^\wedge$ for simplicity. We will show that $P_{BP}(X)_p^\wedge \simeq *$ and that $P_{BP}(Y)_p^\wedge \simeq Y$. Since both nullification and p -completion functors commute with products, the previous splitting $X \simeq B\mathcal{F} \times Y$ shows that $Y \simeq *$. Since $(P_{\Sigma BP}C)_p^\wedge$ is 1-connected, then it follows that it is contractible.

First, we have that $P_{BP}(X)_p^\wedge \simeq P_{BP}(CW_{BP}(B\mathcal{F}))_p^\wedge \simeq *$ by checking the hypothesis of [CF15, Lemma 3.9]. One only needs to check that $P_{BP}(X)$ is p -good and that $P_{BP}(X)_p^\wedge$ is BP -null. Notice that $\pi_1 X$ is a finite p -group since C is 1-connected by Lemma 5.6. Therefore the space $P_{BP}(X)_p^\wedge$ is BP -null by Proposition 5.7 and the space $P_{BP}(X)$ is p -good by Proposition 4.2. Finally, recall that BP -cellular spaces are BP -acyclic ([DF96, 3.B.1]).

Now,

$$P_{BP}(Y)_p^\wedge \simeq P_{BP}(\Omega(P_{\Sigma BP}C)_p^\wedge)_p^\wedge \simeq (\Omega P_{\Sigma BP}((P_{\Sigma BP}C)_p^\wedge))_p^\wedge \simeq (\Omega(P_{\Sigma BP}C)_p^\wedge)_p^\wedge \simeq \Omega(P_{\Sigma BP}C)_p^\wedge = Y$$

where the second equivalence follows from commutation rules [DF96, 3.A.1], the third holds because $(P_{\Sigma BP}C)_p^\wedge$ is ΣBP -null by Miller's theorem [Mil84, Thm 1.5], and the forth is a commutation of taking loops and p -completion [BK72, V.4.6 (ii)].

Summarizing, we have proved that $c: CW_{BP}(B\mathcal{F}) \rightarrow B\mathcal{F}$ is a mod p equivalence. Finally, using Proposition 4.1 we get that c is a weak equivalence since $CW_{BP}(B\mathcal{F})_p^\wedge \simeq B\mathcal{F}$ and $B\mathcal{F}_Q \simeq *$. \square

6 Examples

Let G be a finite group. The situation when G is generated by elements of order p is well studied by R. Flores and R. Foote in [FF11]. We start by giving a simple example where G is not generated by elements of order p^i .

Example 6.1. Let $G = \Sigma_3$, the permutation group of 3 elements. Σ_3 is generated by transpositions, i.e, by elements of order 2, but the Sylow 3-subgroup of Σ_3 is $S = \mathbb{Z}/3$. Therefore, BS is $B\mathbb{Z}/3^r$ -cellular for all $r \geq 1$ and hence $(B\Sigma_3)_3^\wedge$ is $B\mathbb{Z}/3^r$ -cellular for all $r \geq 1$ by Corollary 5.2.

Notice that $B\Sigma_3$ is not $B\mathbb{Z}/3^r$ -cellular for any $r \geq 1$: applying $\text{map}_*(B\mathbb{Z}/3^r, -)$ to the homotopy fibre sequence $B\mathbb{Z}/3 \xrightarrow{Bi} B\Sigma_3 \rightarrow B\mathbb{Z}/2$, we see that Bi is a $B\mathbb{Z}/3^r$ -equivalence for any r , since $\text{map}_*(B\mathbb{Z}/3^r, B\mathbb{Z}/2) \simeq *$. Then $CW_{B\mathbb{Z}/3^r}(B\Sigma_3) \simeq CW_{B\mathbb{Z}/3^r}(B\mathbb{Z}/3) \simeq B\mathbb{Z}/3$.

We are now interested in the study of $CW_{BP}(B\mathcal{F})$ when $S \neq Cl_{\mathcal{F}}(P)$. Our aim is to get tools to identify $P_{BP}(C)_p^\wedge$ in the p -completed Chacholsky's fibration describing $CW_{BP}(B\mathcal{F})$.

Proposition 6.2. Let (S, \mathcal{F}) be a saturated fusion system, \mathcal{L} be the associated centric linking system and let P be a finite p -group. Assume that there is a saturated fusion system (S', \mathcal{F}') , with associated linking system \mathcal{L}' , and a factorization (up to homotopy)

$$\begin{array}{ccccc} |\mathcal{L}| & \xrightarrow{\eta} & B\mathcal{F} & \xlongequal{\quad} & B\mathcal{F} \\ \pi \downarrow & & \pi_p^\wedge \downarrow & & \downarrow r_p^\wedge \\ |\mathcal{L}'| & \xrightarrow{\eta} & B\mathcal{F}' & \xrightarrow{\tilde{r}} & (P_{\Sigma BP}C)_p^\wedge \end{array}$$

such that

- (i) the map \tilde{r} is injective, i.e., $\ker(\tilde{r}) = \{e\}$,
- (ii) and or F , the homotopy fibre of $\pi: |\mathcal{L}| \rightarrow |\mathcal{L}'|$, or F_p , the homotopy fibre of π_p^\wedge , is $B\mathbb{Z}/p$ -acyclic.

Then \tilde{r} is a homotopy equivalence. In particular, $CW_{BP}(B\mathcal{F}) \simeq F_p$.

Proof. We will proceed as follows: first we will construct a map $\tilde{\pi}_p^\wedge: (P_{\Sigma BP}C)_p^\wedge \rightarrow B\mathcal{F}'$ under $B\mathcal{F}$, and then we will prove that $\tilde{\pi}_p^\wedge$ is a homotopy inverse of \tilde{r}_p^\wedge . This will give us that $CW_{BP}(B\mathcal{F})_p^\wedge \simeq F_p$. Finally we will show, in this situation, that $CW_{BP}(B\mathcal{F})$ is p -complete and hence $CW_{BP}(B\mathcal{F}) \simeq F_p$.

- (a) *Construction of $\tilde{\pi}_p^\wedge: (P_{\Sigma BP}C)_p^\wedge \rightarrow B\mathcal{F}'$.* In order to apply Zabrodsky lemma [Dwy96, Proposition 3.4] to the following situation

$$\begin{array}{ccc} CW_{BP}(B\mathcal{F})_p^\wedge & & \\ \downarrow c_p^\wedge & \nearrow \pi_p^\wedge & \\ B\mathcal{F} & \xrightarrow{\quad} & B\mathcal{F}' \\ \downarrow r_p^\wedge & & \\ (P_{\Sigma BP}C)_p^\wedge & & \end{array}$$

we need to check that two assumptions are satisfied:

- $\pi_p^\wedge \circ c_p^\wedge \simeq *$: or equivalently, $\pi_p^\wedge \circ c: CW_{BP}(B\mathcal{F}) \rightarrow B\mathcal{F}'$ is null-homotopic since $B\mathcal{F}'$ is p -complete and $CW_{BP}(B\mathcal{F})$ is p -good.
According to Proposition 5.4 it is sufficient to show that the composite $\pi_p^\wedge \circ c \circ f$ is null-homotopic for all $f \in \text{map}_*(BP, CW_{BP}(B\mathcal{F}))$. But if we postcompose with \tilde{r} , we have $\tilde{r} \circ \pi_p^\wedge \circ c \circ f \simeq r_p^\wedge \circ c \circ f \simeq *$. There exists $\rho \in \text{Hom}(P, S')$ such that $\pi_p^\wedge \circ c \circ f \simeq \Theta' \circ B\rho$, then $\rho(P) \leq \ker(\tilde{r}) = \{e\}$.
- $\text{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega B\mathcal{F}') \simeq *$: since $B\mathcal{F}'$ and $CW_{BP}(B\mathcal{F})$ are p -good space we have $\text{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, B\mathcal{F}') \simeq \text{map}_*(CW_{BP}(B\mathcal{F}), B\mathcal{F}')$ and hence, taking loops,

$$\text{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega B\mathcal{F}') \simeq \text{map}_*(CW_{BP}(B\mathcal{F}), \Omega B\mathcal{F}').$$

The last mapping space is contractible since $\Omega B\mathcal{F}'$ is BP -null (p :mapping space) and $CW_{BP}(B\mathcal{F})$ is BP -acyclic ([DF96, 3.B.1]).

(b) \tilde{r} and $\tilde{\pi}_p^\wedge$ are homotopy inverse. First note that both $Id_{(P_{\Sigma BP}C)_p^\wedge}$ and $\tilde{r} \circ \tilde{\pi}$ factor r_p^\wedge since $Id_{(P_{\Sigma BP}C)_p^\wedge} \circ r_p^\wedge \simeq \tilde{r} \circ \pi \simeq \tilde{r} \circ \tilde{\pi} \circ r_p^\wedge$. In order to show that $\tilde{r} \circ \tilde{\pi}_p^\wedge \simeq Id_{(P_{\Sigma BP}C)_p^\wedge}$, we will apply Zabrodsky lemma to the fibration $CW_{BP}(B\mathcal{F})_p^\wedge \rightarrow B\mathcal{F} \rightarrow (P_{\Sigma BP}C)_p^\wedge$ and the composite map $\tilde{r}_p^\wedge \circ \pi_p^\wedge: B\mathcal{F} \rightarrow (P_{\Sigma BP}C)_p^\wedge$. Then uniqueness up to homotopy of the factorization will give the desired equivalence.

We check that $\text{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$ and $\tilde{r}_p^\wedge \circ \tilde{\pi}_p^\wedge \circ c_p^\wedge \simeq *$. First, by Proposition 5.7, $(P_{\Sigma BP}C)_p^\wedge$ is ΣBP -null, then $\Omega(P_{\Sigma BP}C)_p^\wedge$ is BP -null ([DF96, 3.A.1]). Now, since $CW_{BP}(B\mathcal{F})_p^\wedge$ and $\Omega(P_{\Sigma BP}C)_p^\wedge$ are p -good and $CW_{BP}(B\mathcal{F})$ is BP -acyclic, we have

$$\text{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq \text{map}_*(CW_{BP}(B\mathcal{F}), \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$$

Finally, $\tilde{r}_p^\wedge \circ \tilde{\pi}_p^\wedge \circ c_p^\wedge \simeq r_p^\wedge \circ c_p^\wedge \simeq *$.

It remains to prove that $\tilde{\pi}_p^\wedge \circ \tilde{r}_p^\wedge \simeq Id_{B\mathcal{F}'}$. First note that $Id_{B\mathcal{F}'} \circ \pi \simeq \tilde{\pi} \circ r_p^\wedge \simeq \tilde{\pi} \circ \tilde{r} \circ \pi$, then both $Id_{B\mathcal{F}'}$ and $\tilde{\pi} \circ \tilde{r}$ factor π . In order to show that $Id_{B\mathcal{F}'} \simeq \tilde{\pi} \circ \tilde{r}$, we will apply again Zabrodsky's lemma to the fibration $F_p \rightarrow B\mathcal{F} \rightarrow B\mathcal{F}'$ (resp. $F \rightarrow |\mathcal{L}| \rightarrow |\mathcal{L}'|$, depending on whether F or F_p is $B\mathbb{Z}/p$ -acyclic) and the composite $\tilde{\pi} \circ r_p^\wedge$ (resp. $\tilde{\pi} \circ r_p^\wedge \circ \eta$).

If $i: F_p \rightarrow B\mathcal{F}$, then $r_p^\wedge \circ i \simeq \tilde{r} \circ \pi \circ i \simeq *$. Then since $\Omega B\mathcal{F}'$ is $B\mathbb{Z}/p$ -null and F_p is $B\mathbb{Z}/p$ -acyclic (resp. F is $B\mathbb{Z}/p$ -acyclic), $\text{map}_*(F_p, \Omega B\mathcal{F}') \simeq *$ (resp. $\text{map}_*(F, \Omega B\mathcal{F}') \simeq *$).

(c) $CW_{BP}(B\mathcal{F})$ is p -complete. By Proposition 4.1 we obtain the homotopy fibration

$$CW_{BP}(B\mathcal{F}) \rightarrow F_p \rightarrow (F_p)_Q,$$

where $(F_p)_Q \simeq *$, since $\pi_i(F_p)$ are finite groups for $i \geq 1$.

□

Remark 6.3. Proposition 6.2 also holds without assuming the existence of π and only considering the factorization on p -completed classifying spaces if we know that the homotopy fiber F_p is $B\mathbb{Z}/p$ -acyclic.

Corollary 6.4. *Let (S, \mathcal{F}) be a saturated fusion system and let P be a finite p -group. If $\mathcal{O}_{\mathcal{F}}^p(S) \triangleleft Cl_{\mathcal{F}}(P)$, then $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'$, where $B\mathcal{F}'$ is the connected cover of $B\mathcal{F}$ with $\pi_1 B\mathcal{F}' \cong Cl_{\mathcal{F}}(P)/\mathcal{O}_{\mathcal{F}}^p(S)$.*

Proof. The connected cover $B\mathcal{F}'$ is the classifying space of a saturated fusion system over $Cl_{\mathcal{F}}(P)$ by [BCG⁺07, Theorem A]. So $B\mathcal{F}'$ is BP -cellular by Corollary 5.2. We get the following commutative diagram of homotopy fibrations

$$\begin{array}{ccccc} B\tilde{\mathcal{F}} & \xlongequal{\quad} & B\tilde{\mathcal{F}} & & \\ \downarrow & & \downarrow & & \\ B\mathcal{F}' & \xrightarrow{\quad f \quad} & B\mathcal{F} & \longrightarrow & B(S/Cl_{\mathcal{F}}(P)) \\ \downarrow & & \downarrow & & \parallel \\ B(Cl_{\mathcal{F}}(P)/\mathcal{O}_{\mathcal{F}}^p(S)) & \longrightarrow & B(S/\mathcal{O}_{\mathcal{F}}^p(S)) & \longrightarrow & B(S/Cl_{\mathcal{F}}(P)) \end{array}$$

In order to apply Proposition 6.2, we have to find a factorization of $r_p^\wedge: B\mathcal{F} \rightarrow (P_{\Sigma BP}C)_p^\wedge$ by a map $\tilde{r}: B(S/Cl_{\mathcal{F}}(P)) \rightarrow (P_{\Sigma BP}C)_p^\wedge$ with trivial kernel, where we can assume that C is 1-connected by step 2 in the proof of Proposition 4.2.

By Lemma 5.3 and Remark 3.3, $\Omega(P_{\Sigma BP}C)_p^\wedge$ is BP -null and $B\mathcal{F}'$ is BP -cellular space (hence BP acyclic), then $\text{map}_*(B\mathcal{F}, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$. Moreover, $r_p^\wedge \circ f \simeq *$, because $r_p^\wedge \circ f|_{B(Cl_{\mathcal{F}}(P))} \simeq *$ since $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$. We are in the conditions of applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) which shows that there is a map $\tilde{r}: B(S/Cl_{\mathcal{F}}(P)) \rightarrow (P_{\Sigma BP}C)_p^\wedge$ such that the diagram

$$\begin{array}{ccc} B\mathcal{F}' & & \\ \downarrow f & & \\ B\mathcal{F} & \xlongequal{\quad} & B\mathcal{F} \\ \downarrow & & \downarrow r_p^\wedge \\ B(S/Cl_{\mathcal{F}}(P)) & \xrightarrow{\tilde{r}} & (P_{\Sigma BP}C)_p^\wedge \end{array}$$

commutes up to homotopy.

Let $[x] \in \ker(\tilde{r}) \leq S/Cl_{\mathcal{F}}(P)$, where $x \in S$. Since the above diagram commutes, $r_p^\wedge|_{B\langle x \rangle} \simeq \tilde{r}|_{B\langle [x] \rangle} \simeq *$, then $x \in \ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$ and hence $[x] = e$. Finally, Proposition 6.2 gives us the equivalence $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'$. \square

Corollary 6.5. *Let (S, \mathcal{F}) be a fusion system and let P be a finite p -group. If $Cl_{\mathcal{F}}(P) \triangleleft \mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is homotopy equivalent to the homotopy fibre of $B\mathcal{F} \rightarrow B(\mathcal{F}/Cl_{\mathcal{F}}(P))$.*

Proof. Let $K = Cl_{\mathcal{F}}(P)$. Since K is normal in \mathcal{F} , there is a saturated fusion system $(S/K, \mathcal{F}/K)$, and a map defined between the nerve of the associated linking systems $\pi: |\mathcal{L}| \rightarrow |\mathcal{L}/K|$ whose homotopy fibre is BK , and an injective factorization of r_p^\wedge by $B\mathcal{F}/K$ from Proposition 3.8. Then the result follows from Proposition 6.2. \square

Example 6.6. Let G be a finite group and let S be a p -Sylow subgroup of G . Assume that $N_G(S)$ controls fusion in G . Then $BN_G(S)_p^\wedge \simeq BG_p^\wedge$ and S is normal in $\mathcal{F}_S(N_G(S))$. On account of Corollary 6.5, for all finite p -group P , $CW_{BP}(BG_p^\wedge)$ is equivalent to the homotopy fibre of $BN_G(S)_p^\wedge \rightarrow B(N_G(S)/Cl_{\mathcal{F}_S(N_G(S))}(P))_p^\wedge$.

An example is given by $G = \mathbb{Z}/p^n \wr \mathbb{Z}/q = (\mathbb{Z}/p^n)^q \rtimes \mathbb{Z}/q$, when $p \neq q$ and $n \geq 1$. The Sylow p -subgroup of G is $S = (\mathbb{Z}/p^n)^q$ and $Cl_{\mathcal{F}_S(G)}(\mathbb{Z}/p^r) = (\mathbb{Z}/p^r)^q$ is abelian and hence normal in $\mathcal{F}_S(G)$ by Corollary 3.9. Therefore BG_p^\wedge is $B\mathbb{Z}/p^r$ -cellular if and only if $r \geq n$ by Theorem 5.1. Then $CW_{B\mathbb{Z}/p^r}(BG_p^\wedge)$ is equivalent to the homotopy fibre of $BG_p^\wedge \rightarrow B(G/(\mathbb{Z}/p^r)^q)_p^\wedge$ by Corollary 6.5.

Other explicit examples appear in [FS07, Example 5.2]. The authors proved that the normalizer of the Sylow of the Suzuki group $Sz(2^n)$, with n an odd integer at least 3, is $N_{Sz(2^n)}(S) = S \rtimes \mathbb{Z}/(2^n - 1)$ and it controls fusion in $Sz(2^n)$. In this case, S is $B\mathbb{Z}/2^m$ -cellular for all $m \geq 2$ and hence $BSz(2^n)_2^\wedge$ is so. Moreover, $Cl_{\mathcal{F}_S(Sz(2^n))}(\mathbb{Z}/2) \cong (\mathbb{Z}/2)^n$ and hence $CW_{B\mathbb{Z}/2}(BSz(2^n)_2^\wedge)$ is equivalent to the homotopy fibre of $BN_{Sz(2^n)}(S)_2^\wedge \rightarrow B(N_{Sz(2^n)}(S)/(\mathbb{Z}/2)^n)_2^\wedge$.

7 The cellularization of the classifying spaces of a family of exotic fusion systems at $p = 3$

Let $S = B(3, r; 0, \gamma, 0)$, for $r \geq 4$ and $\gamma = 0, 1, 2$, be the family of finite 3-groups of order 3^r (see [DRV07, Theorem A.2, Proposition A.9]) generated by $\{s, s_1, \dots, s_{r-1}\}$ where

- $s_i = [s_{i-1}, s]$ for all $i \in \{2, \dots, r-1\}$,
- $[s_1, s_i] = 1$ for all $i \in \{2, \dots, r-1\}$,
- $s_1^3 s_2^3 s_3 = s_{r-1}^\gamma$,
- $s_i^3 s_{i+1}^3 s_{i+2} = 1$ for all $i \in \{2, \dots, r-1\}$,
- $s^3 = 1$.

The center of S is $\langle s_{r-1} \rangle$. The normal subgroup $\gamma_1 = \langle s_1, \dots, s_{r-1} \rangle$ of S is of index 3 and the corresponding group extension is split. There are group isomorphisms

$$B(3, r; 0, \gamma, 0) = \langle s_1, s_2 \rangle \rtimes \langle s \rangle = \begin{cases} (\mathbb{Z}/3^m \times \mathbb{Z}/3^m) \rtimes \mathbb{Z}/3 & , \text{ if } r = 2m + 1, \\ (\mathbb{Z}/3^m \times \mathbb{Z}/3^{m+1}) \rtimes \mathbb{Z}/3 & , \text{ if } r = 2m. \end{cases}$$

In [DRV07, Theorem 5.10], the authors construct families of exotic 3-local finite groups \mathcal{F} whose Sylow 3-subgroup is $S = B(3, r; 0, \gamma, 0)$.

Proposition 7.1. *Let \mathcal{F} be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ such that \mathcal{F} has at least one \mathcal{F} -Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then*

- (i) *If $\gamma = 0$, then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$ -cellular for all $l \geq 1$.*
- (ii) *Assume $\gamma \neq 0$. Then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$ -cellular if and only if $l \geq 2$. If $l = 1$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = \langle s, s_2 \rangle$.*

Proof. By Theorem 5.1 we are reduced to the computation of $Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$. Let $N := \langle s_2, s \rangle$ which is a proper normal subgroup of S . This subgroup N contains the center of S , $Z(S) = \langle s_{r-1} \rangle \cong \mathbb{Z}/3$ by [DRV07, Lemma A.10]. Moreover, the description of \mathcal{F} and the computation of automorphisms of S in [DRV07, Theorem 5.10, Lemma A.14] show that N is strongly \mathcal{F} -closed.

First $s \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ for all $l \geq 1$. If $Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ is a proper normal subgroup of S , again by [DRV07, Lemma A.10], $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = N$ with $[N : S] = 3$, and

$$N \cong \begin{cases} 3_+^{1+2} & , \text{ if } r = 4, \\ B(3, r-1; 0, 0, 0) & , \text{ if } r > 4. \end{cases}$$

We have then inclusions $N \subset Cl_{\mathcal{F}}(\mathbb{Z}/3^l) \subset S$. And $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ if and only if there exists $x \in S \setminus N$ such that $x^{3^l} = 1$.

Let $x = s^i s_1^j s_2^k \in S$, where $i = 0, 1, 2$. Since the index $[Cl_{\mathcal{F}}(\mathbb{Z}/3^l) : S] \in \{1, 3\}$, $x \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ if and only if $x^2 \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$. But if $x = s^2 s_1^j s_2^k$, then $x^2 = s^4 s_1^{j'} s_2^{k'} = s s_1^{j'} s_2^{k'}$, therefore we

can assume $i = 0, 1$. Note that $s_i = [s_{i-1}, s] \in \langle s, s_2 \rangle \subset N$ for all $i \in \{3, \dots, r-1\}$ and $s_1^3 = s_{r-1}^\gamma s_3^{-1} s_2^{-3} \in N$. If $j \equiv 0 \pmod 3$ then $x \in N$.

If $i = 0$ then $x = s_1^j s_2^k$ and $[s_1, s_2] = 1$. Then $o(x) = 3$ if and only if $o(s_1^j) = 3$ and $o(s_2^k) = 3$. In particular, $3j|3^m$, that is, $3|j$. But then $j \equiv 0 \pmod 3$ and $x \in N$.

If $i = 1$ and $x = s s_1^j s_2^k$, one can compute $x^3 = (s_{r-1})^{\gamma i}$ using relations $s_1^a s = s s_1^a s_2^a$, $s_2^b = s s_2^b s_3^b$, $s_3^c s = s s_3^c s_4^c$, $s_1^3 s_2^3 s_3 = s_{r-1}^\gamma$ and $s_2^3 s_3^3 s_4 = 1$ with $a, b, c > 0$ ([DRV07, Proposition A.9]). We have that $x^{3^l} = (s_{r-1})^{\gamma i 3^{l-1}}$ and $x^{3^l} = 1$ if and only if $\gamma i 3^{l-1} \equiv 0 \pmod 3$. That is, $x^9 = 1$ always, and $x^3 = 1$ if $\gamma = 0$ or $i \equiv 0 \pmod 3$, but in this last case $x \in N$.

Summarizing, if $\gamma = 0$ then $x = s s_1 \in S \setminus N$ is of order 3 and then $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ for all $l \geq 1$. If $\gamma \neq 0$ then $x = s s_1 \in S \setminus N$ is of order 9 and then $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ for all $l \geq 2$. Finally if $\gamma \neq 0$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = N$. \square

We will finish this section by describing $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ when \mathcal{F} is an exotic fusion system over $B(3, r; 0, \gamma, 0)$ with $\gamma \neq 0$ such that \mathcal{F} has at least one \mathcal{F} -Alperin rank two elementary abelian 3-subgroup given in [DRV07].

Lemma 7.2. *Let \mathcal{F} an exotic fusion system under the hypothesis of Proposition 7.1, $B\mathcal{F}$ is 1-connected.*

Proof. By Proposition 2.5, the fundamental group of $B\mathcal{F}$ is $\pi_1(B\mathcal{F}) \cong S/\mathcal{O}_{\mathcal{F}}^p(S)$, where $\mathcal{O}_{\mathcal{F}}^p(S) := \langle [Q, \mathcal{O}^p(\text{Aut}_{\mathcal{F}}(Q))] \mid Q \leq S \rangle$. The subgroup $\mathcal{O}_{\mathcal{F}}^p(S)$ is a strongly \mathcal{F} -closed subgroup of S , and the arguments in the proof in [DRV07, page 1751] show that it must contain $N = \langle s, s_2 \rangle < S$. We will show that $\mathcal{O}_{\mathcal{F}}^p(S) = S$ by proving that $s_1 \in \mathcal{O}_{\mathcal{F}}^p(S)$. Checking tables in [DRV07, Theorem 5.10, Lemma A.14], we see that the automorphisms of order two η and/or ω are group elements in $\text{Aut}_{\mathcal{F}}(S)$. By the description given there $\eta(s_1) = s_1 s_2^{f''}$ and $\omega(s_1) = s_1^{-1} s_2^{f''}$. Then $s_1^{-1} \eta(s_1) = s_1^{-2} s_2^{f''}$ or $s_1^{-1} \omega(s_1) = s_1^{-2} s_2^{f''}$ are elements of $\mathcal{O}_{\mathcal{F}}^p(S)$, since $s_2, s_1^3 \in N \subset \mathcal{O}_{\mathcal{F}}^p(S)$ then $s_1 \in \mathcal{O}_{\mathcal{F}}^p(S)$. \square

Proposition 7.3. *Let \mathcal{F} be an exotic fusion system satisfying the hypothesis of Proposition 7.1 with $\gamma \neq 0$. Let $N = \langle s, s_2 \rangle < S$, then there exists a unique map (up to homotopy) $f: B\mathcal{F} \rightarrow (B\Sigma_3)_3^\wedge$ whose kernel is N .*

Proof. The proof of Proposition 3.7 shows that the quotient morphism $S \rightarrow S/N \cong \mathbb{Z}/3$ gives a fusion preserving homomorphism $\rho: S \rightarrow \Sigma_3$. We want to show that this morphism extends to a map $f: B\mathcal{F} \rightarrow (B\Sigma_3)_3^\wedge$.

By Proposition 2.9, $B\mathcal{F} \simeq (\text{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B}P)_3^\wedge$, where $\tilde{B}P \simeq BP$ for $P \in \mathcal{F}^c$. The fusion preserving property of ρ shows that $B\rho \in \lim_{\mathcal{O}(\mathcal{F})} [BP, (B\Sigma_3)_3^\wedge]$.

The obstructions for rigidifying the homotopy commutative diagram in the category of spaces lie in $\lim_{\mathcal{O}^c(\mathcal{F})}^{i+1} \pi_i(\text{map}(BP, (B\Sigma_3)_3^\wedge)_{\Theta' \circ B\rho|_P})$, for $i \geq 1$ (see [Woj87]). Note that since the 3-Sylow subgroup of Σ_3 is abelian, we have $\pi_1(\text{map}(BP, (B\Sigma_3)_3^\wedge)_{\Theta' \circ B\rho|_P})$ is abelian being a quotient of $C_{\mathbb{Z}/3}(\rho(P))$. In fact, it will be trivial or $\mathbb{Z}/3$ (Proposition 2.6).

We will show that for any $F: \mathcal{O}(\mathcal{F}) \rightarrow \mathbb{Z}_{(p)}\text{-Mod}$, $\lim_{\mathcal{O}(\mathcal{F})}^i F = 0$ for $i > 1$. From [BLO03b, Proposition 3.2, Corollary 3.3], we are reduced to show that derived limits of atomic functors have the same vanishing property. Note that from [DRV07, Theorem

5.10, Lemma A.14], the relevant automorphism groups $\text{Out}_{\mathcal{F}}(P)$ are $SL_2(\mathbb{F}_3)$ or $GL_2(\mathbb{F}_3)$. In both cases the 3-Sylow subgroup is of order 3, and then [JMO92, Proposition 6.2(i)] implies the result.

The obstructions to uniqueness lie in $\lim_{O^c(\mathcal{F})}^i \pi_i(\text{map}(BP, (B\Sigma_3)_3^\wedge)^{\Theta' \circ B\rho|_P})$, for $i \geq 1$ (see [Woj87]). By the previous paragraph we have to look at the first derived functor of atomic functors with value $\mathbb{Z}/3$. But since $\text{Aut}(\mathbb{Z}/3) \cong \mathbb{Z}/2$, by [JMO92, Proposition 6.1(ii)] any element of order 3 will act trivially on $\mathbb{Z}/3$. Finally note that if a map $g: B\mathcal{F} \rightarrow (B\Sigma_3)_3^\wedge$ has kernel N , its restriction $g|_{BS}: BS \rightarrow (B\Sigma_3)_3^\wedge$ has to be homotopic to $B\rho$. \square

Proposition 7.4. *Let \mathcal{F} be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ with $\gamma \neq 0$ such that \mathcal{F} has at least one \mathcal{F} -Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then there exists a map $f: B\mathcal{F} \rightarrow (B\Sigma_3)_3^\wedge$ such that $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ is the homotopy fiber of f .*

Proof. Let f be the map constructed in Proposition 7.3 with $\ker(f) = Cl_{\mathcal{F}}(S)$. Precisely because of this, $f \circ ev \simeq *$ where $ev: \bigvee_{[B\mathbb{Z}/3, B\mathcal{F}]} B\mathbb{Z}/3 \rightarrow B\mathcal{F}$. Then f factors through the cofibre C of ev and, since $(B\Sigma_3)_3^\wedge$ is $B\mathbb{Z}/3$ -null, we obtain a factorization of f , $f': P_{\Sigma B\mathbb{Z}/3}(C) \rightarrow (B\Sigma_3)_3^\wedge$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} B\mathcal{F} & \xlongequal{\quad} & B\mathcal{F} \\ \downarrow & & \downarrow r_3^\wedge \\ P_{B\mathbb{Z}/3}(C) & \xrightarrow{f'} & (B\Sigma_3)_3^\wedge. \end{array}$$

The strategy is to construct a homotopy inverse of f' , $\bar{\Theta}: (B\Sigma_3)_3^\wedge \rightarrow P_{B\mathbb{Z}/3}(C)_3^\wedge$, up to 3-completion, which fits in the previous diagram up to homotopy.

Since Σ_3 has an abelian normal 3-Sylow subgroup $\mathbb{Z}/3$, we have that $(B\mathbb{Z}/3)_{h\mathbb{Z}/2} \rightarrow B\Sigma_3$ is an equivalence. Consider the fibre sequence $BN \rightarrow BS \rightarrow B\mathbb{Z}/3$ and the map $r_3^\wedge|_{BS}: BS \rightarrow P_{B\mathbb{Z}/3}(C)_3^\wedge$, by Zabrodsky's lemma [Dwy96, Proposition 3.4], $r_3^\wedge|_{BS}$ factors (uniquely up to homotopy) via $\Theta': B\mathbb{Z}/3 \rightarrow P_{B\mathbb{Z}/3}(C)_3^\wedge$. In order to get $\bar{\Theta}$, we only need check that Θ' is $\mathbb{Z}/2$ -equivariant up to homotopy. For any \mathcal{F} in the hypothesis of the proposition, note that there is an element in $\omega' \in \text{Out}_{\mathcal{F}}(S)$ which project to $\omega \in \text{Out}_{\Sigma_3}(\mathbb{Z}/3)$ (they are called η or ω in the tables [DRV07, Theorem 5.10]). Since Θ' is unique up to homotopy factoring $r_3^\wedge \circ \Theta$, and $\Theta \circ \omega' \simeq \Theta$, it follows $\omega \circ \Theta' \simeq \Theta'$.

Next we check that $\bar{\Theta}$ is a homotopy inverse to f' . First consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} CW_{B\mathbb{Z}/3}(B\mathcal{F}) & & & & \\ \downarrow & & & & \\ B\mathcal{F} & \xlongequal{\quad} & B\mathcal{F} & \xlongequal{\quad} & B\mathcal{F} \\ \downarrow r & & \downarrow & & \downarrow r_3^\wedge \\ P_{\Sigma B\mathbb{Z}/3}(C) & \xrightarrow{f'} & (B\Sigma_3)_3^\wedge & \xrightarrow{\bar{\Theta}} & P_{\Sigma B\mathbb{Z}/3}(C)_3^\wedge. \end{array}$$

Since 3-completion on the bottom line also gives a homotopy commutative diagram, unicity on Zabrodsky's lemma (see [Dwy96, Proposition 3.4]) shows that $\bar{\Theta} \circ f'$ is 3-completion. So $\bar{\Theta} \circ (f')_3^\wedge \simeq id$.

Now consider the following homotopy commutative diagram

$$\begin{array}{ccccc}
BK & \longrightarrow & F & & \\
\downarrow & & \downarrow & & \\
BS & \xrightarrow{\Theta} & B\mathcal{F} & \equiv & B\mathcal{F} \\
\downarrow & & \downarrow & & \downarrow f \\
B\mathbb{Z}/3 & \xrightarrow{\iota} & (B\Sigma_3)_3^\wedge & \xrightarrow{(f')_3^\wedge \circ \Theta} & (B\Sigma_3)_3^\wedge.
\end{array}$$

The map $(f')_3^\wedge \circ \Theta$ is determined by its restriction to the Sylow 3-subgroup $\mathbb{Z}/3$ by Proposition 7.3. Again $\iota \circ (f')_3^\wedge \circ \Theta$ and ι give homotopy commutative diagrams when placed in the bottom line, then unicity on Zabrodsky's lemma (see [Dwy96, Proposition 3.4]) shows that they are homotopic. \square

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